Mathematics 110: Lecture 14

Independent Bernoulli Trials

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February 22, 2019

Bernoulli trials

- We call an experiment with exactly two possible outcomes a *Bernoulli trial*.
- Generically, we often refer to the two outcomes as either *success* (S) or *failure* (F).
- Examples:
  - Tossing a coin and recording a success for heads and a failure for tails.
  - Rolling a die and recording a success for rolling a 1 and a failure for rolling anything else.
  - Testing the lifetime of a light bulb and recording a success if the light bulb lasts at least 1000 hours and a failure otherwise.
- We will let p be the probability of a success in a Bernoulli trial and q be the probability of a failure.
- Note: $p + q = 1$ and $q = 1 - p$.
- A *Bernoulli process* is a sequence of independent repetitions of the same Bernoulli trial.

Example

- Suppose we roll a fair die 5 times and count the number of times we roll an ace.
- This is a Bernoulli process with $p = \frac{1}{6}$ and $q = \frac{5}{6}$.
- Q: What is the probability we obtain exactly 3 aces?
- Let S denote rolling an ace and F denote rolling any other number.
  - *SSFSF* is one particular sequence with 3 successes.
  - The probability of this particular sequence is

$$\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} = \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2.$$

- Note:
  - We would have the same probability for any particular sequence with three successes and two failures.
  - The number of such sequences is $C(5,3)$ since we need to choose three locations out of a possible five for the successes.

Example (cont'd)

- Hence if E is the event that there are exactly 3 successes, then

$$\Pr(E) = C(5,3)\left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 = 0.03215.$$

Repeated Bernoulli trials

- Suppose we repeat the same Bernoulli trial, independently, n times.
- Let p be the probability of success and q be the probability of failure on each trial.
- Q: What is the probability of exactly r successes?
- Note:
  - Any sequence of n trials with exactly r successes and $n-r$ failures will have probability
$$p^r q^{n-r}.$$
  - There are $C(n,r)$ such sequences.
- Consequently, the probability of exactly r successes is
$$C(n,r)p^r q^{n-r}.$$

Example

- Suppose a fair coin is tossed 20 times.
- The probability of exactly 10 heads is
$$C(20,10)\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{10} = 184,756 \times \left(\frac{1}{2}\right)^{20} = \frac{184,756}{1,048,576} = 0.1762.$$

Example

- A English lady claims she can tell the order in which the tea and milk were added to a cup.
- Ten cups of tea are prepared for her, in which some were randomly selected to have the milk poured before the tea.
- The lady is the asked to taste each cup and identify which ones had the milk poured first.
- Suppose she identifies seven of the cups correctly.
- Q: What is the probability that she would identify seven or more cups correctly if she is merely guessing?
- Note: If she is guessing, the probability of a success (that is, identifying a cup correctly) is 0.5.

Example (cont'd)

- So the probability of identifying seven or more cups correctly would be
$$\begin{aligned}
&C(10,7)\left(\frac{1}{2}\right)^{7}\left(\frac{1}{2}\right)^{3} + C(10,8)\left(\frac{1}{2}\right)^{8}\left(\frac{1}{2}\right)^{2} + C(10,9)\left(\frac{1}{2}\right)^{9}\left(\frac{1}{2}\right)^{1} \\
&\quad + C(10,10)\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{0} \\
&= 120\left(\frac{1}{2}\right)^{10} + 45\left(\frac{1}{2}\right)^{10} + 10\left(\frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{10} \\
&= \frac{176}{1024} = 0.1719.
\end{aligned}$$
- Note:
  - So she has an 17% chance of identifying 7 or more with just guessing.
  - Thus we would not consider this sufficient evidence to indicate that she really has the ability she claims.

Example

- A drug used to treat a certain disease has a success rate of 70%.
- Suppose a new drug is tested on 20 patients, and is successful on 18.
- Q: What is the probability that the new drug would cure 18 or more patients if it is no more effective than the old drug?
- If the new drug has the same effectiveness as the old, then this is a Bernoulli process with $n = 20$ and $p = 0.7$.
- So the probability of 18 or more succeses would be

$$\begin{aligned} C(20,18)(0.7)^{18}(0.3)^2 + C(20,19)(0.7)^{19}(0.3)^1 + C(20,20)(0.7)^{20}(0.3)^0 \\ &= 190(0.7)^{18}(0.3)^2 + 20(0.7)^{19}(0.3) + (0.7)^{20} \\ &= 0.03548. \end{aligned}$$

- Since this probability is so small, this would be considered evidence in favor of the new drug being better than the old drug for treating this disease.