Mathematics 110: Lecture 21

Two Linear Equations in Two Unknowns

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Example

- A bakery sells two types of chocolate chip cookies:
  - The large cookies use 3 ounces of dough and 1 ounce of chocolate chips.
  - The small cookies use 2 ounces of dough and 0.5 ounces of chocolate chips.
- Suppose the bakery has 700 ounces of dough and 200 ounces of chocolate chips to work with each day.
- Q: How many cookies of each type should the bakery make in order to exactly use up the dough and the chips?
- Solution:
  - Let x be the number of large cookies made and y the number of small cookies made.
  - Amount of dough used: $3x + 2y$.
  - Amount of chips used: $x + 0.5y$.

Example (cont'd)

- Solution (cont'd):
  - So we want the equations

$$3x + 2y = 700$$
$$x + 0.5y = 200$$

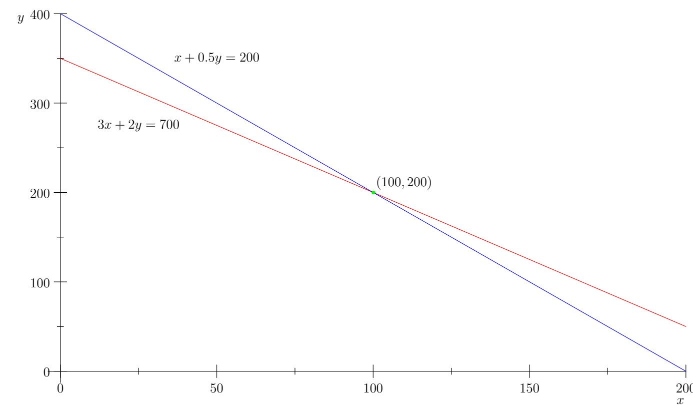
  to be true simultaneously.
  - From the second equation we have $x = 200 - 0.5y$.
  - Putting this value of x in the first equation, we have

$$700 = 3(200 - 0.5y) + 2y = 600 - 1.5y + 2y = 600 + 0.5y.$$

  - Hence $0.5y = 100$.
  - And so $y = 2 \times 100 = 200$.
  - Then $x = 200 - (0.5 \times 200) = 200 - 100 = 100$.
  - Thus the bakery should make 100 large cookies and 200 small cookies.

Example

- Graphs of $3x + 2y = 700$ and $x + 0.5y = 200$:



Two linear equations in two variables

- Consider two linear equations with variables x and y:

$$ax + by = e$$
$$cx + dy = f.$$

- Case I:
  - The equations might represent the same line, in which case there are an infinite number of solutions.
  - Example: The equations

$$2x + 4y = 10$$
$$x + 2y = 5$$

are both equations for the line $y = -\frac{1}{2}x + \frac{5}{2}$.

Two linear equations in two variable (cont'd)

- Case II:
  - The equations might represent distinct lines which are parallel, in which case there are no solutions.
  - We say such an system of equations is *inconsistent*.
  - Example: The equations

$$2x + 4y = 10$$
$$x + 2y = 6$$

are equations of the distinct parallel lines $y = -\frac{1}{2}x + \frac{5}{2}$ and $y = -\frac{1}{2}x + 3$.

Two linear equations in two variables (cont'd)

- Case III:
  - The equations might represent non-parallel lines, in which case there is exactly one solution (where the two lines intersect).
  - Example: Consider the equations

$$4x - 2y = 10$$
$$x + 3y = 1.$$

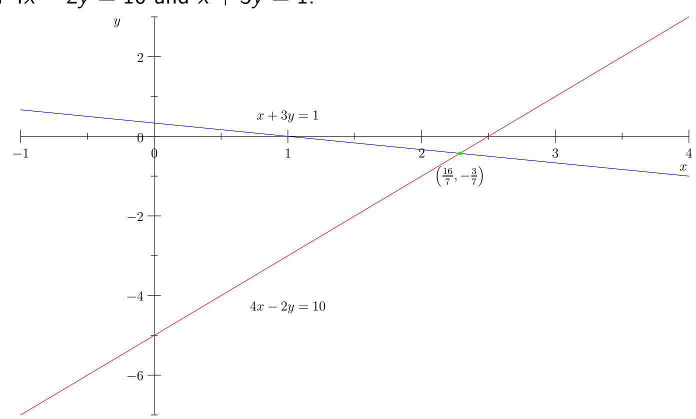
  - From the second equation we have $x = 1 - 3y$.
  - Then the first equation becomes

$$10 = 4(1 - 3y) - 2y = 4 - 12y - 2y = 4 - 14y.$$

  - Hence $14y = -6$, so $y = -\frac{6}{14} = -\frac{3}{7}$.
  - When $y = -\frac{3}{7}$, $x = 1 + \left(3 \times \frac{3}{7}\right) = \frac{16}{7}$.
  - That is, the two lines intersect at the point $\left(\frac{16}{7}, -\frac{3}{7}\right)$.

Two linear equations in two variables (cont'd)

- Graphs of $4x - 2y = 10$ and $x + 3y = 1$:



Reduction vs. substitution

- Substitution method of solving equations:
  - Solve for one of the variables in one of the equations.
  - Substitute this value in the other equation.
  - Now solve the resulting equation (which has only one variable)
- Reduction method for solving equations:
  - Allowed operations:
    - Multiply an equation by a non-zero constant.
    - Replace a given equation by the sum of that equation and a multiple of the other equation.
  - Repeat these operations until one, or both, of the equations is either of the form x equal to a constant or y equal to a constant.

Example

- Consider the equations

$$\begin{aligned} 2x + 4y &= 6 \\ 3x - 2y &= -7. \end{aligned}$$

- Multiply the first equation by $\frac{1}{2}$:

$$\begin{aligned} x + 2y &= 3 \\ 3x - 2y &= -7. \end{aligned}$$

- Add -3 times the first equation to the second equation:

$$\begin{aligned} x + 2y &= 3 \\ -8y &= -16. \end{aligned}$$

Example (cont'd)

- Multiply the second equation by $-\frac{1}{8}$:

$$\begin{aligned} x + 2y &= 3 \\ y &= 2. \end{aligned}$$

- We may now substitute $y = 2$ in the first equation to obtain $x = 3 - 4 = -1$.
- Or add -2 times the second equation to the first equation:

$$\begin{aligned} x &= -1 \\ y &= 2. \end{aligned}$$

Example

- Consider the system of equations

$$\begin{aligned} 2x - 4y &= 10 \\ -3x + 6y &= -15. \end{aligned}$$

- Multiply the first equation by $\frac{1}{2}$:

$$\begin{aligned} x - 2y &= 5 \\ -3x + 6y &= -15. \end{aligned}$$

- Add 3 times the first equation to the second equation:

$$\begin{aligned} x - 2y &= 5 \\ 0 &= 0. \end{aligned}$$

- It follows that there are an infinite number of solutions, namely, $y = \frac{1}{2}x - \frac{5}{2}$ for any value of x.

Example

- Consider the system of equations

$$\begin{aligned} 2x - 4y &= 12 \\ -3x + 6y &= -15. \end{aligned}$$

- As in the previous example, multiply the first equation by $\frac{1}{2}$:

$$\begin{aligned} x - 2y &= 6 \\ -3x + 6y &= -15. \end{aligned}$$

- And add 3 times the first equation to the second equation:

$$\begin{aligned} x - 2y &= 5 \\ 0 &= 3. \end{aligned}$$

- But this time the second equation is impossible, and so the system is inconsistent, that is, has no solutions.