Mathematics 110: Lecture 26

The Leontief Input-Output Model

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Example

- Suppose a country produces exactly two goods: wood and steel.
- Suppose the production of one unit of wood requires 0.2 units of wood and 0.3 units of steel.
- Suppose the production of one unit of steel requires 0.4 units of wood and 0.6 units of steel.
- In addition, suppose there is an external demand for 4 units of wood and 2 units of steel.
- Let x_1 be the amount of wood produced and x_2 be the amount of steel produced.
- Then we must have

$$\begin{aligned} x_1 &= 0.2x_1 + 0.4x_2 + 4 \\ x_2 &= 0.3x_1 + 0.6x_2 + 2. \end{aligned}$$

Example (cont'd)

- Let
$$A = \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$
- Then we may write the above system of equations as
$$X = AX + D.$$
- It follows that $D = X - AX = IX - AX = (I - A)X$.
- Hence
$$X = (I - A)^{-1}D.$$

Example (cont'd)

- Now
$$I - A = \begin{bmatrix} 0.8 & -0.4 \\ -0.3 & 0.4 \end{bmatrix}.$$
- And
$$\left[\begin{array}{cc|cc} 0.8 & -0.4 & 1 & 0 \\ -0.3 & 0.4 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & -0.5 & 1.25 & 0 \\ -0.3 & 0.4 & 0 & 1 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{cc|cc} 1 & -0.5 & 1.25 & 0 \\ 0 & 0.25 & 0.375 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & -0.5 & 1.25 & 0 \\ 0 & 1 & 1.5 & 4 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 2 & 2 \\ 0 & 1 & 1.5 & 4 \end{array}\right]$$

Example (cont'd)

- Hence $(I - A)^{-1} = \begin{bmatrix} 2 & 2 \\ 1.5 & 4 \end{bmatrix}$.
- So we have
$$X = \begin{bmatrix} 2 & 2 \\ 1.5 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 14 \end{bmatrix}.$$
- Thus the country should produce $x_1 = 12$ units of wood and $x_2 = 14$ units of steel.

Leontief input-output model

- Consider an economy which has n industries, each of which produces a single good.
- Let $x_1, x_2, \ldots, x_n$ be the amount of each good produced for a specified time.
- Let A be a matrix for which the entry in the ith row and jth column is the amount of good i needed to produce one unit of of good j.
- We call A the *technology*, or *input-output*, matrix.
- Let $d_1, d_2, \ldots, d_n$ be the external demands for the goods.
- We call
$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$
the *production schedule* and the *external demand vector*, respectively.

Leontief input-output model

- It then follows that
$$X = AX + D.$$
- If $I - A$ is invertible, we then find that
$$X = (I - A)^{-1}D.$$
- We call the above the *Leontief input-output model*.
- Note: The solution shows how we can gauge the effect of changing demand vectors on the production levels of the economy.

Example

- Leontief analyzed the American economy in 1958, using a simplified model which divided the economy into 81 sectors.
- We will look at a further simplified model which consolidates the data into three sectors: industry (I), energy (E), and services (S).
- For these sectors, the following table states the amount of input of each the sectors (in millions of dollars) required to produce one unit of output (again, in millions of dollars) for the each of the three sectors:

		Output		
		I	**E**	**S**
	I	0.474	0.026	0.079
Input	**E**	0.018	0.358	0.025
	S	0.105	0.173	0.234

- For example, one unit of industrial output requires 0.474 units of industrial input, 0.018 units of energy input, and 0.105 units of service input.

Example (cont'd)

- In other words, the input-output matrix was

$$A = \begin{bmatrix} 0.474 & 0.026 & 0.079 \\ 0.018 & 0.358 & 0.025 \\ 0.105 & 0.173 & 0.234 \end{bmatrix}.$$

- The external demands were: 223,133 for industrial output, 23,527 for energy output, and 263,985 for service output, all in millions of dollars.
- That is, the external demand vector was

$$D = \begin{bmatrix} 223{,}133 \\ 23{,}527 \\ 263{,}985 \end{bmatrix}.$$

Example (cont'd)

- Now if x_1 is the amount of industrial output, x_2 the amount of energy output, and x_3 the amount of services output, then

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is the desired production schedule.
- And we want to find $X = (I - A)^{-1}D$.
- Now

$$I - A = \begin{bmatrix} 0.526 & -0.026 & -0.079 \\ -0.018 & 0.642 & -0.025 \\ -0.105 & -0.173 & 0.766 \end{bmatrix}$$

Example (cont'd)

- And

$$(I - A)^{-1} = \begin{bmatrix} 1.947 & 0.134 & 0.205 \\ 0.066 & 1.576 & 0.058 \\ 0.282 & 0.374 & 1.347 \end{bmatrix}$$

- Hence

$$X = (I - A)^{-1}D = \begin{bmatrix} 491{,}709 \\ 67{,}116 \\ 427{,}310 \end{bmatrix}.$$