Mathematics 140: Lecture 25

More on the Chain Rule

Dan Sloughter

Furman University

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The chain rule in functional notation

- Suppose h and g are differentiable.
- Let $f(x) = h \circ g(x) = h(g(x))$.
- If we let $y = h(u)$ and $u = g(x)$, then

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = h'(u)g'(x) = h'(g(x))g'(x).$$

Example

- Let $f(x) = (x^2 + 1)^{10}$.
- Then $f(x) = h(g(x))$, where

$$h(x) = x^{10} \text{ and } g(x) = x^2 + 1.$$

- Now

$$h'(x) = 10x^9 \text{ and } g'(x) = 2x.$$

- So

$$f'(x) = h'(g(x))g'(x) = 10(x^2 + 1)^9(2x) = 20x(x^2 + 1)^9.$$

Special case

- Suppose $f(x) = (g(x))^n$.
- Then $f(x) = h(g(x))$ where $h(x) = x^n$.
- Now $h'(x) = nx^{n-1}$.
- So

$$f'(x) = h'(g(x))g'(x) = n(g(x))^{n-1}g'(x).$$

Example

- Suppose $f(x) = (x^3 + 2x - 1)^{20}$.
- Then

$$\begin{aligned} f'(x) &= 20(x^3 + 2x - 1)^{19}(3x^2 + 2) \\ &= 20(3x^2 + 2)(x^3 + 2x - 1)^{19}. \end{aligned}$$

Example

- Suppose $g(t) = \dfrac{4}{\sqrt{t^2 + 8}}$.
- Then

$$\begin{aligned} g'(t) &= -2(t^2 + 8)^{-\frac{3}{2}}(2t) \\ &= -\frac{4t}{(t^2 + 8)^{\frac{3}{2}}}. \end{aligned}$$

Example

- Suppose $g(x) = x^4\sqrt{x^2 + 10}$.
- Then

$$\begin{aligned} g'(x) &= x^4\left(\frac{1}{2}(x^2 + 10)^{-\frac{1}{2}}(2x)\right) + 4x^3\sqrt{x^2 + 10} \\ &= \frac{x^5}{\sqrt{x^2 + 10}} + 4x^3\sqrt{x^2 + 10}. \end{aligned}$$

Example

- Suppose $h(t) = \dfrac{(t^4 + 3t^2 + 1)^{10}}{(t^2 + 4)^5}$.
- Then

$$\begin{aligned} &h'(t) = \\ &\frac{(t^2 + 4)^5(10(t^4 + 3t^2 + 1)^9(4t^3 + 6t)) - (t^4 + 3t^2 + 1)^{10}(5(t^2 + 4)^4(2t))}{(t^2 + 4)^{10}} \\ &= \frac{10(4t^3 + 6t)(t^2 + 4)^5(t^4 + 3t^2 + 1)^9 - 10t(t^2 + 4)^4(t^4 + 3t^2 + 1)^{10}}{(t^2 + 4)^{10}} \\ &= \frac{(t^2 + 4)^4(t^4 + 3t^2 + 1)^9\left(10(4t^3 + 6t)(t^2 + 4) - 10t(t^4 + 3t^2 + 1)\right)}{(t^2 + 4)^{10}} \\ &= \frac{10(t^4 + 3t^2 + 1)^9(3t^5 + 19t^3 + 24t - 1)}{(t^2 + 4)^6} \end{aligned}$$

Example

- Suppose $f(x) = \frac{10}{3x+4}$.
- Then
$$f'(x) = -10(3x+4)^{-2}(3) = -\frac{30}{(3x+4)^2}.$$

Example

- Suppose $g(x) = f(x^2)$.
- Then
$$g'(x) = 2xf'(x^2).$$