Mathematics 145: Lecture 21

Derivatives of Exponential Functions

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A limit

- An important limit:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

Derivative of e^x

- Suppose $f(x) = e^x$.
- Then

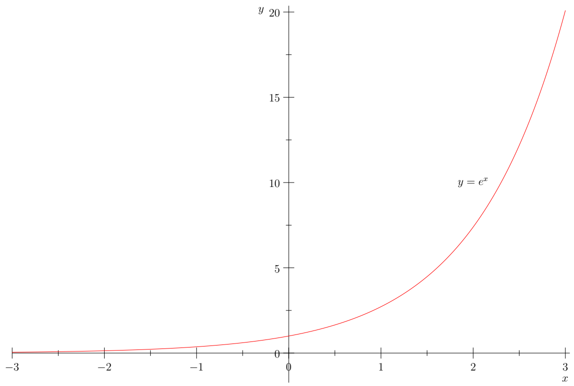
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \\ &= e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x. \end{aligned}$$

Example

- Let $f(x) = e^x$.
- Then $f'(x) = e^x$, so $f(x) > 0$ for all x.
- That is, the graph of f is increasing on $(-\infty, \infty)$.
- Moreover, $f''(x) = e^x$, so $f''(x) > 0$ for all x.
- That is, the graph of f is concave upward on $(-\infty, \infty)$.

Example (cont'd)

- Graph of $y = e^x$:

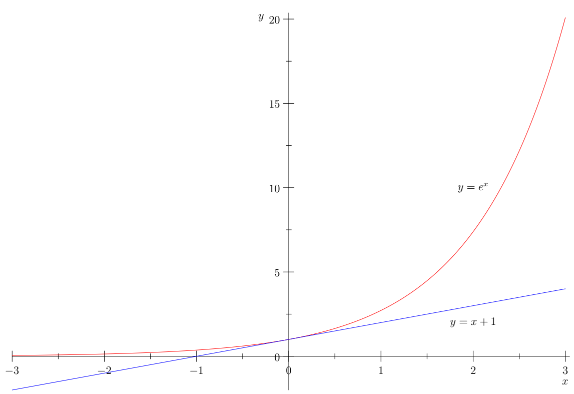


Example

- Q: What is the equation of the line tangent to the graph of $y = e^x$ at $(0, 1)$?
- Since $\frac{dy}{dx} = e^x$,
$$\left.\frac{dy}{dx}\right|_{x=0} = e^0 = 1.$$
- So the equation of the tangent line is
$$y = x + 1.$$

Example (cont'd)

- Graph of $y = e^x$ with tangent line at $(0, 1)$:



Examples

- Example:
  - Suppose $f(x) = e^{4x}$.
  - Then $f'(x) = 4e^{4x}$.
- Example:
  - Suppose $f(x) = 5e^{-4x^2}$.
  - Then $f'(x) = -40xe^{-4x^2}$.
- Example:
  - Suppose $f(x) = x^2e^{-x}$.
  - Then
$$f'(x) = x^2(-e^{-x}) + 2xe^{-x} = -x^2e^{-x} + 2xe^{-x} = (2x - x^2)e^{-x}.$$

Example

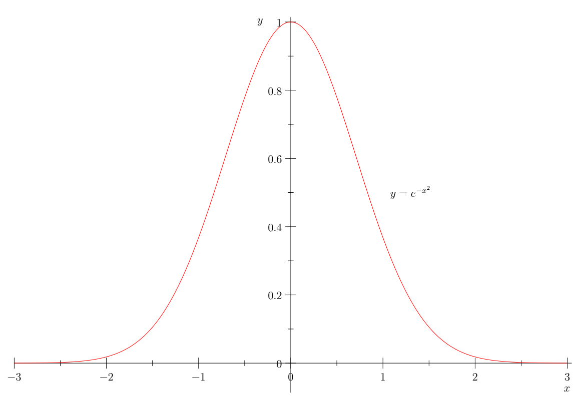
- Let $f(x) = e^{-x^2}$.
- Then $f'(x) = -2xe^{-x^2}$.
- So $f'(x) = 0$ when $x = 0$.
- And
  - $f'(-1) = 2e^{-1} > 0$,
  - and $f'(1) = -2e^{-1} < 0$.
- So f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.
- Hence f has a relative maximum of 1 at $x = 0$.
- Then $f''(x) = -2x(-2xe^{-x^2}) - 2e^{-x^2} = (4x^2 - 2)e^{-x^2}$.
- So $f''(x) = 0$ when $4x^2 - 2 = 0$, that is, when $x^2 = \frac{1}{2}$.
- So $f''(x) = 0$ when $x = -\frac{1}{\sqrt{2}}$ and when $x = \frac{1}{\sqrt{2}}$.

Example (cont'd)

- Now
  - $f''(-1) = 2e^{-1} > 0$,
  - $f''(0) = -2 < 0$,
  - and $f''(1) = 2e^{-1} > 0$.
- So the graph of f is concave upward on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and on $\left(\frac{1}{\sqrt{2}}, \infty\right)$, and concave downward on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.
- Thus the graph of f has inflection points at $\left(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$.
- Also,
$$\lim_{x\to-\infty} e^{-x^2} = 0 \text{ and } \lim_{x\to\infty} e^{-x^2} = 0.$$
- So the x-axis is a horizontal asymptote for the graph of f.

Example (cont'd)

- Graph of $y = e^{-x^2}$:



Example

- Suppose $f(x) = \frac{1}{1+e^{-x}}$.
- Then
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{1+e^{-x}} = \frac{1}{1+0} = 1.$$
- And
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{1+e^{-x}} = 0.$$
- Hence the lines $y = 1$ and $y = 0$ are both horizontal asymptotes for the graph of f.

Example (cont'd)

- Graph of $y = \frac{1}{1+e^{-x}}$: