Mathematics 145: Lecture 6

Limits

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Example

- Let $f(x) = \frac{\sqrt{x^2+4}-2}{x^2}$.
- Note: f is not defined when $x = 0$.
- However, we may evaluate f for values of x arbitrarily close to 0.
- For example:

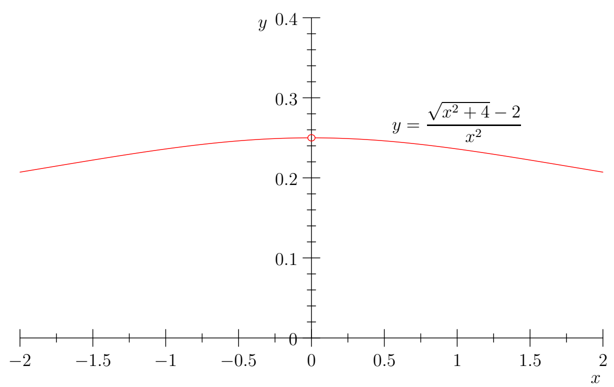
x	-0.05	-0.1	-0.001	0.001	0.01	0.05
$f(x)$	0.246211	0.249843	0.249998	0.249998	0.249843	0.246211

- When we look at the graph, it looks like there is some value L to which $f(x)$ approximates arbitrarily closely when we take x sufficiently close to 0.
- If there is such a value, we would write

$$\lim_{x \to 0} \frac{\sqrt{x^2+4}-2}{x^2} = L.$$

Example (cont'd)

- Graph of $y = \frac{\sqrt{x^2+4}-2}{x^2}$:



Idea of a limit

- Suppose f is a function and c and L are real numbers.
- We say the *limit* of $f(x)$ as x approaches c is L if it is possible to make $f(x)$ arbitrarily close to L by taking x sufficiently close to c.
- Note:
  - In this case, we write $\lim_{x \to c} f(x) = L$.
  - c may be in the domain of f, but does not have to be in the domain of f.

Two important cases

- If f is a polynomial, then $\lim_{x \to c} f(x) = f(c)$ for any real number c.
- If f is a rational function and c is in the domain of f, then $\lim_{x \to c} f(x) = f(c)$

Properties

- If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then
  - $\lim_{x \to c} (f(x) + g(x)) = L + M$,
  - $\lim_{x \to c} (f(x) - g(x)) = L - M$,
  - $\lim_{x \to c} kf(x) = kL$ for any constant k,
  - $\lim_{x \to c} f(x)g(x) = LM$,
  - $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, provided $M \neq 0$,
  - $\lim_{x \to c} (f(x))^p = L^p$ for any p, provided L^p is defined.

Examples

- $\lim_{x \to 4} (x^2 + 4) = 16 + 4 = 20$
- $\lim_{x \to 5} \frac{x^2 + 3x}{4x + 1} = \frac{40}{21}$
- $\lim_{x \to 3} \left(\sqrt{x+1} - \frac{1}{\sqrt{x+6}} \right) = 2 - \frac{1}{3} = \frac{5}{3}$
- $\lim_{x \to 4} \frac{x+1}{x-4}$ does not exist because the denominator is approaching 0 while the numerator is approaching 5.

Property

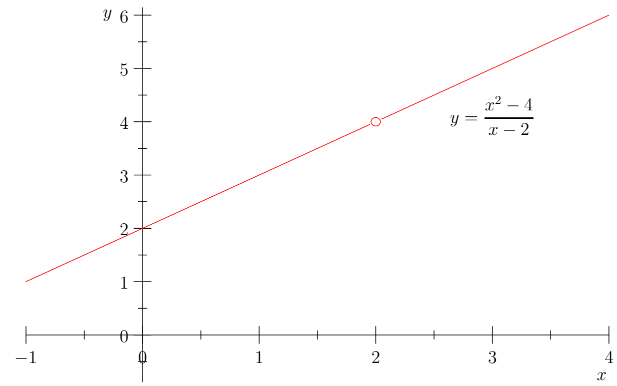
- Suppose $\lim_{x \to c} f(x) = L$.
- Suppose $g(x) = f(x)$ for all $x \neq c$.
- Then $\lim_{x \to c} g(x) = \lim_{x \to c} f(x) = L$

Example

- Since
$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$$
whenever $x \neq 2$, we have
$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

Example (cont'd)

- Graph of $y = \frac{x^2-4}{x-2}$:



Examples

- $\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$
- $\lim_{x\to 4} \frac{x^2-x-12}{x^2-2x-8} = \lim_{x\to 4} \frac{(x-4)(x+3)}{(x-4)(x+2)} = \lim_{x\to 4} \frac{x+3}{x+2} = \frac{7}{6}$
- $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} = \lim_{x\to 9}(\sqrt{x}+3) = 6$
- $\lim_{x\to 1} \frac{x^2-1}{x^2-2x+1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)^2} = \lim_{x\to 1} \frac{x+1}{x-1}$, so the limit does not exist.