Mathematics 150: Lecture 15

Implicit Differentiation

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Implicit differentiation

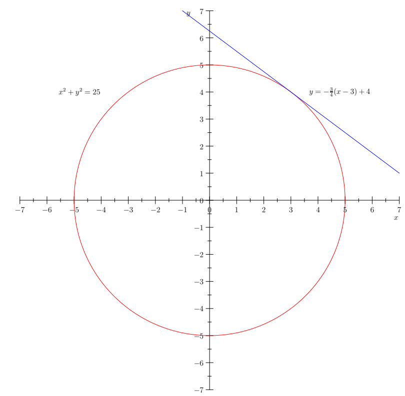
- The technique used to find the derivative of a rational power is useful in finding the slope of a curve defined by an equation, but not explicitly expressed as the graph of a function. We call this *implicit differentiation*.

Example

- Consider the equation
$$x^2 + y^2 = 25.$$
- The graph of this equation is the circle C of radius 5 centered at the origin.
- Suppose we wish to find the slope of C at a point, say, the point $(3, 4)$.
- One way to proceed would be to solve for y explicitly in terms of x and then differentiate the resulting expression.
- Note: such an explicit expression would work for only some points on C since, overall, C is not the graph of a function.

Example (cont'd)

- Graph of $x^2 + y^2 = 25$ with tangent line at $(3, 4)$:



Example (cont'd)

- Another approach: differentiate both sides of the expression,

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25),$$

to obtain

$$2x + 2y\frac{dy}{dx} = 0.$$

- Note: since we are differentiating with respect to x, we treated y as a function of x and used the chain rule to differentiate y^2.
- Now we may solve for $\frac{dy}{dx}$, obtaining

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Example (cont'd)

- Evaluating at our point of interest, we have

$$\left.\frac{dy}{dx}\right|_{(x,y)=(3,4)} = -\frac{3}{4}.$$

- So the equation of the line tangent to C at $(3, 4)$ is

$$y = -\frac{3}{4}(x - 3) + 4.$$

- Note: our expression for $\frac{dy}{dx}$ is valid only when $y \neq 0$.
- The lines tangent to C at the two points where $y = 0$, namely, $(-5, 0)$ and $(5, 0)$ are vertical.
- Hence we should not expect to find derivatives at those points.

Example

- Let C be the curve with equation

$$y^5 - 3xy^2 + 3x^2 = 7.$$

- To find $\frac{dy}{dx}$, we first differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(y^5 - 3xy^2 + 3x^2) = \frac{d}{dx}7.$$

- That is,

$$5y^4\frac{dy}{dx} - 3x\left(2y\frac{dy}{dx}\right) - 3y^2 + 6x = 0.$$

- Hence

$$5y^4\frac{dy}{dx} - 6xy\frac{dy}{dx} - 3y^2 + 6x = 0.$$

- So

$$\frac{dy}{dx}(5y^4 - 6xy) = 3y^2 - 6x.$$

Example

- Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{3y^2 - 6x}{5y^4 - 6xy}, \text{ provided } 5y^4 - 6xy \neq 0.$$

- For example, $(2, 1)$ is on C, and

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{3 - 12}{5 - 12} = \frac{9}{7}.$$

- So the equation of the line tangent to C at $(2, 1)$ is

$$y = \frac{9}{7}(x - 2) + 1.$$

Example (cont'd)

- The curve $y^5 - 3xy^2 + 3x^2 = 7$ with tangent line at $(2, 1)$: