Mathematics 150: Lecture 31

Change of Variables: Indefinite Integrals

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Example: reversing the chain rule

- Suppose we wish to evaluate
$$\int 2x\sqrt{1+x^2}dx.$$
- Since $2x$ is the derivative of $1+x^2$ and
$$\int \sqrt{x}dx = \frac{2}{3}x^{\frac{3}{2}} + c,$$
we might guess
$$\int 2x\sqrt{1+x^2}dx = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c,$$
which we may verify by differentiation.

Example

- Note: If we wanted to evaluate
$$\int x\sqrt{1+x^2}dx$$
we might still guess
$$\frac{2}{3}(1+x^2)^{\frac{3}{2}}.$$
- However,
$$\frac{d}{dx}\left(\frac{2}{3}(1+x^2)^{\frac{3}{2}}\right) = 2x\sqrt{1+x^2}.$$

Example (cont'd)

- We could correct our guess by multiplying by $\frac{1}{2}$.
- That is,
$$\int x\sqrt{1+x^2}dx = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c,$$
which we may check by differentiation.

Substitution

- Note: The first example above is of the form
$$\int f(g(x))g'(x)dx,$$
where $f(x) = \sqrt{x}$ and $g(x) = 1 + x^2$.
- In general, if F is an antiderivative of f, then, by the chain rule,
$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

Leibniz notation

- Using Leibniz notation, if $u = g(x)$, then
$$\int f(u)\frac{du}{dx}dx = F(u) + c.$$
- But $\int f(u)du = F(u) + c$, so we have
$$\int f(\underbrace{g(x)}_{u})\underbrace{g'(x)dx}_{du} = \int f(u)\frac{du}{dx}dx = \int f(u)du.$$
- Note: Symbolically (and only symbolically), we may think of substituting u for $g(x)$ and du for $g'(x)dx$.

Example

- To evaluate $\int 2x\sqrt{1+x^2}dx$, we make the substitution
$$u = 1 + x^2,$$
from which we have
$$\frac{du}{dx} = 2x \text{ or } du = 2xdx.$$
- Hence
$$\int 2x\sqrt{1+x^2}dx = \int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c.$$

Example

- To evaluate $\int x\sqrt{1+x^2}dx$, we make the substitution
$$u = 1 + x^2$$
$$du = 2xdx,$$
from which it follows that $\frac{1}{2}du = xdx$.
- Then we have
$$\int x\sqrt{1+x^2}dx = \frac{1}{2}\int \sqrt{u}du = \frac{1}{3}u^{\frac{3}{2}} + c = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c,$$
as we saw above.

Example

- To evaluate $\int x^2 \sin(4x^3)dx$, we make the substitution

$$u = 4x^3$$
$$du = 12x^2 dx, \text{ or } \frac{1}{12}du = x^2 dx.$$

- Then

$$\int x^2 \sin(4x^3)dx = \frac{1}{12}\int \sin(u)du = -\frac{1}{12}\cos(u) + c = -\frac{1}{12}\cos(4x^3) + c.$$

Example

- To evaluate $\int \sin^2(x)\cos(x)dx$, we make the substitution

$$u = \sin(x)$$
$$du = \cos(x)dx.$$

- Then

$$\int \sin^2(x)\cos(x)dx = \int u^2 du = \frac{1}{3}u^3 + c = \frac{1}{3}\sin^3(x) + c.$$

Example

- To evaluate $\int \sqrt{4x+5}dx$, we make the substitution

$$u = 4x + 5$$
$$du = 4dx, \text{ or } \frac{1}{4}du = dx.$$

- Then

$$\int \sqrt{4x+5}dx = \frac{1}{4}\int \sqrt{u}du = \frac{2}{12}u^{\frac{3}{2}} + c = \frac{1}{6}(4x+5)^{\frac{3}{2}} + c.$$

Example

- To evaluate $\int x\sqrt{1+x}dx$, we make the substitution

$$u = 1 + x$$
$$du = dx.$$

- Then, using $x = u - 1$,

$$\int x\sqrt{1+x}dx = \int (u-1)\sqrt{u}du = \int u^{\frac{3}{2}}du - \int \sqrt{u}du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + c.$$

- Note: Substitution worked in this problem, although it did not involve reversing the chain rule.