Mathematics 150: Lecture 33

Area Between Curves

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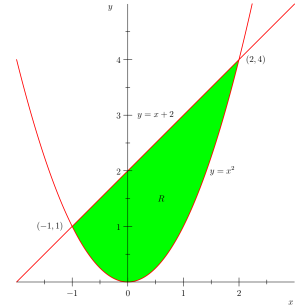
Example: area between curves

- Let A be the area of the region R bounded by the curves $y = x^2$ and $y = x + 2$:



- Now $x^2 = x + 2$ when $0 = x^2 - x - 2 = (x-2)(x+1)$, that is, when $x = -1$ or $x = 2$.
- Hence the two curves intersect at the points $(-1, 1)$ and $(2, 4)$.

Example (cont'd)

- Now for $-1 \leq x \leq 2$, R is the region bounded above by the curve $y = x + 2$ and below by the curve $y = x^2$.
- Since

$$A_1 = \int_{-1}^{2} (x+2)dx$$

is the area beneath $y = x + 2$ and above $[-1, 2]$ and

$$A_2 = \int_{-1}^{2} x^2 dx$$

is the area beneath $y = x^2$ and above $[-1, 2]$, $A = A_1 - A_2$.
- That is,

$$A = \int_{-1}^{2} (x+2)dx - \int_{-1}^{2} x^2 dx = \int_{-1}^{2} (2 + x - x^2)dx.$$

Example (cont'd)

- Hence

$$\begin{aligned} A &= 2x\Big|_{-1}^{2} + \frac{1}{2}x^2\Big|_{-1}^{2} - \frac{1}{3}x^3\Big|_{-1}^{2} \\ &= (4+2) + \left(2 - \frac{1}{2}\right) - \left(\frac{8}{3} + \frac{1}{3}\right) \\ &= 5 - \frac{1}{2} \\ &= \frac{9}{2} \end{aligned}$$

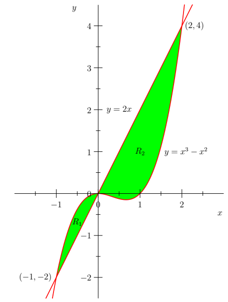
Area between curves

- In general, if $f(x) \le g(x)$ for all x in the interval $[a, b]$ and A is the area of the region R which lies between the curves $y = f(x)$ and $y = g(x)$ over the interval $[a, b]$, then

$$A = \int_a^b (g(x) - f(x))dx.$$

Example

- Let A be the area of the region R bounded by the curves $y = x^3 - x^2$ and $y = 2x$:



- First we see that the two curves intersect when $x^3 - x^2 = 2x$, that is, when

$$0 = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x - 2)(x + 1).$$

Example (cont'd)

- Hence the curves intersect at $(-1, -2)$, $(0, 0)$ and $(2, 4)$.
- Now for $-1 < x < 0$, $x^3 - x^2 > 2x$, and for $0 < x < 2$, $x^3 - x^2 < 2x$.
- Break R into two regions:
  - R_1, bounded above by $y = x^3 - x^2$ and below by $y = 2x$ for $-1 \le x \le 0$,
  - R_2, bounded above by $y = 2x$ and below by $y = x^3 - x^2$ for $0 \le x \le 2$.
- Hence

$$\begin{aligned} A &= \int_{-1}^{0} (x^3 - x^2 - 2x)dx + \int_0^2 (2x - x^3 + x^2)dx \\ &= \frac{1}{4}x^4\Big|_{-1}^{0} - \frac{1}{3}x^3\Big|_{-1}^{0} - x^2\Big|_{-1}^{0} + x^2\Big|_0^2 - \frac{1}{4}x^4\Big|_0^2 + \frac{1}{3}x^3\Big|_0^2 \\ &= -\frac{1}{4} - \frac{1}{3} + 1 + 4 - 4 + \frac{8}{3} = \frac{37}{12}. \end{aligned}$$

Example

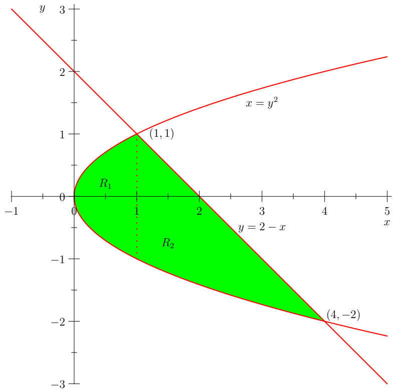
- We will find the area A of the region R bounded by the curves $x = y^2$ and $y = 2 - x$.
- Note: these curves intersect when $y^2 = 2 - y$, that is when

$$0 = y^2 + y - 2 = (y + 2)(y - 1).$$

- Thus the curves intersect at the points $(1, 1)$ and $(4, -2)$.
- Note: over the interval $[0, 1]$, R is bounded above by $y = \sqrt{x}$ and below by $y = -\sqrt{x}$ and, over the interval $[1, 4]$, R is bounded above by $y = 2 - x$ and below by $y = -\sqrt{x}$.

Example (cont'd)

- The region between the curves $x = y^2$ and $y = 2 - x$:



Example (cont'd)

- Hence we have

$$\begin{aligned} A &= \int_0^1 (\sqrt{x} - (-\sqrt{x}))dx + \int_1^4 (2 - x - (-\sqrt{x}))dx \\ &= 2\int_0^1 \sqrt{x}dx + \int_1^4 (2 - x + \sqrt{x})dx \\ &= \frac{4}{3}x^{\frac{3}{2}}\Big|_0^1 + 2x\Big|_1^4 - \frac{1}{2}x^2\Big|_1^4 + \frac{2}{3}x^{\frac{3}{2}}\Big|_1^4 \\ &= \frac{4}{3} + (8 - 2) - \left(8 - \frac{1}{2}\right) + \left(\frac{16}{3} - \frac{2}{3}\right) \\ &= \frac{9}{2}. \end{aligned}$$

Another approach

- Note: if A is the area of the region bounded by the curves $x = g(y)$ and $x = f(y)$ over an interval $[c, d]$, where we assume $f(y) \leq g(y)$ for all y in $[c, d]$, then, analogous to our previous formula,

$$A = \int_c^d (g(y) - f(y))dy.$$

Example (cont'd)

- We may also find the area A of the region R by

$$\begin{aligned} A &= \int_{-2}^1 (2 - y - y^2)dy \\ &= 2y\Big|_{-2}^1 - \frac{1}{2}y^2\Big|_{-2}^1 - \frac{1}{3}y^3\Big|_{-2}^1 \\ &= (2 + 4) - \left(\frac{1}{2} - 2\right) - \left(\frac{1}{3} + \frac{8}{3}\right) \\ &= \frac{9}{2}. \end{aligned}$$

Example

- Let A be the area of the region R bounded by $x = y^2$ and $x = y$.
- Now $y = y^2$ when $y(y-1) = 0$, that is, when $y = 0$, or $y = 1$.
- Hence the curves intersect at $(0, 0)$, and $(1, 1)$.
- Thus

$$A = \int_0^1 (y - y^2)dy = \frac{1}{2}y^2\Big|_0^1 - \frac{1}{3}y^3\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Example (cont'd)

- Note: we could also find A as follows:

$$A = \int_0^1 (\sqrt{x} - x)dy = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 - \frac{1}{2}x^2\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

Example (cont'd)

- Region R bounded by the curves $x = y^2$ and $x = y$: