Mathematics 250: Lecture 30

Green's Theorem

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Simple regions

- A *region of Type III* is a region in $\mathbb{R}^2$ which is both of Type I and of Type II.
- B is *simple region* if it is of Type III and its boundary, ∂B, is a piecewise smooth curve.
- Suppose B is a simple region, $F(x,y) = (p(x,y), q(x,y))$, and p and q have continuous first-order partial derivatives on an open set which contains B.
- Since B is of Type I, there are functions $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ and constants a and b such that

$$B = \{(x,y) : a \le x \le b, \varphi(x) \le y \le \psi(x)\}.$$

Simple regions (cont'd)

- With ∂B oriented in the counter-clockwise direction, $\partial B = C_1 + C_2 + C_2 + C_4$, where

$$\alpha_1(t) = (t, \varphi(t)), a \le t \le b,$$

parametrizes C_1,

$$\alpha_2(t) = (b, t), \varphi(b) \le t \le \psi(b),$$

parametrizes C_2,

$$\alpha_3(t) = (t, \psi(t)), a \le t \le b,$$

parametrizes $-C_3$, and

$$\alpha_4(t) = (a, t), \varphi(a) \le t \le \psi(a),$$

parametrizes $-C_4$.

Simple regions (cont'd)

- Then

$$\int_{\partial B} p dx = \int_{C_1} p dx + \int_{C_2} p dx - \int_{-C_3} p dx - \int_{-C_4} p dx.$$

- And

$$\int_{C_1} p dx = \int_a^b (p(t, \varphi(t)), 0) \cdot (1, \varphi'(t)) dt = \int_a^b p(t, \varphi(t)) dt,$$

$$\int_{C_2} p dx = \int_{\varphi(b)}^{\psi(b)} (p(b,t), 0) \cdot (0,1) dt = 0,$$

$$\int_{-C_3} p dx = \int_a^b (p(t, \psi(t)), 0) \cdot (1, \psi'(t)) dt = \int_a^b p(t, \psi(t)) dt,$$

and

$$\int_{-C_4} p dx = \int_{\varphi(a)}^{\psi(a)} (p(a,t), 0) \cdot (0,1) dt = 0.$$

Simple regions (cont'd)

- Hence

$$\begin{aligned}\int_{\partial B} p dx &= \int_a^b (p(t,\varphi(t)) - p(t,\psi(t)))dt \\ &= -\int_a^b \int_{\varphi(t)}^{\psi(t)} \frac{\partial}{\partial y} p(t,y) dy dt \\ &= -\int_a^b \int_{\varphi(x)}^{\psi(x)} \frac{\partial}{\partial y} p(x,y) dy dx \\ &= -\int\int_B \frac{\partial}{\partial y} p(x,y)\ dA.\end{aligned}$$

- A similar calculation, treating B as a region of Type II, gives us

$$\int_{\partial B} q dy = \int\int_B \frac{\partial}{\partial x} q(x,y)\ dA.$$

Green's theorem for simple regions

- Putting these results together, we have

$$\int_{\partial B} p dx + q dy = \int\int_B \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\ dA,$$

which is *Green's theorem*, or *Stoke's theorem*, for a simple region.

Example

- Suppose B is the region in the plane bounded by the triangle with vertices at $(0,0)$, $(2,0)$, and $(0,3)$.
- Then

$$\begin{aligned}\int_{\partial B} (3x^2 + y)dx + 5x dy &= \int\int_B \left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(3x^2 + y)\right)\ dA \\ &= \int\int_B (5-1)\ dA \\ &= 4\int\int_B dA \\ &= 4 \times 3 \\ &= 12,\end{aligned}$$

where we have used the fact that the triangle B has area 3.

Area

- Note: If p and q are such that

$$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 1,$$

then

$$\int_{\partial B} p dx + q dy = \int\int_B dA = \text{ area of } B.$$

- For example, if A is the area of B, then

$$A = \int_{\partial B} x dy,$$

$$A = -\int_{\partial B} y dx,$$

and

$$A = \frac{1}{2}\int_{\partial B} x dy - y dx.$$

Example

- Let B be the region bounded by the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$
- Then $\alpha(t)=(a\cos(t), b\sin(t))$, $0\le t\le 2\pi$, parametrizes ∂B.
- Hence, if A is the area inside the ellipse,
$$\begin{aligned} A &= \frac{1}{2}\int_{\partial B} xdy - ydx \\ &= \frac{1}{2}\int_0^{2\pi}(-b\sin(t), a\cos(t))\cdot(-a\sin(t), b\cos(t))dt \\ &= \frac{1}{2}\int_0^{2\pi}(ab\sin^2(t)+ab\cos^2(t))dt \\ &= \frac{ab}{2}\int_0^{2\pi} dt = ab\pi. \end{aligned}$$

Example

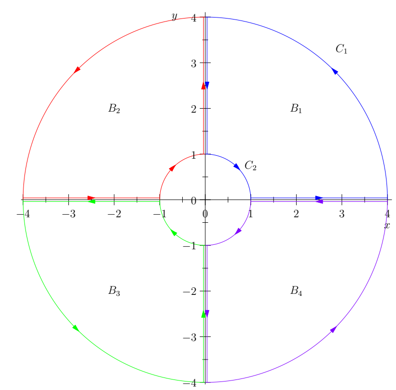
- Let $B=\{(x,y): 1\le x^2+y^2\le 16\}$.
- That is, B is the annular region between the circle C_1 with equation $x^2+y^2=16$ and the circle C_2 with equation $x^2+y^2=1$.
- Note:
  - B is not a simple region.
  - However, B is the union of the simple regions
$$\begin{aligned} B_1 &= \{(x,y): 1\le x^2+y^2\le 16, x\ge 0, y\ge 0\}, \\ B_2 &= \{(x,y): 1\le x^2+y^2\le 16, x\le 0, y\ge 0\}, \\ B_3 &= \{(x,y): 1\le x^2+y^2\le 16, x\le 0, y\le 0\}, \\ B_4 &= \{(x,y): 1\le x^2+y^2\le 16, x\ge 0, y\le 0\}. \end{aligned}$$

Example (cont'd)

- Hence if $F(x,y)=(p(x,y),q(x,y))$, where p and q have continuous partial derivatives on an open set containing B,
$$\begin{aligned} \int\int_B\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dA &= \int\int_{B_1}\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dA \\ &+ \int\int_{B_2}\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dA \\ &+ \int\int_{B_3}\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dA \\ &+ \int\int_{B_4}\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dA. \end{aligned}$$

Example (cont'd)

- Decomposition of B into $B_1\cup B_2\cup B_3\cup B_4$:



Example (cont'd)

- Using Green's theorem, each of these integrals may be replaced by the line integral of F around the boundaries of B_1, B_2, B_3, and B_4, respectively.
- Note:
  - The integrals along the borders between these region will cancel out (they are each traversed twice, in opposite directions).
  - We are left with the integral around C_1 in the counterclockwise direction and the integral around C_2 in the clockwise direction.
  - With these orientations, we let $\partial B = C_1 + C_2$.
  - C_1 and C_2 are oriented so that, if we were to walk around the boundary of B, the interior of B is always to our left.
  - We call this the *positive* orientation of ∂B.
- Using the positive orientation of ∂B, we now have that

$$\int_{\partial B} p dx + q dy = \int \int_B \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) \, dA.$$

Example (cont'd)

- For example,

$$\begin{aligned}
\int_{\partial B} (3x^2 + y)dx + 5x dy &= \int \int_B \left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(3x^2 + y) \right) \, dA \\
&= \int \int_B (5 - 1) \, dA \\
&= 4 \int \int_B dA \\
&= 4 \times (16\pi - \pi) \\
&= 60\pi.
\end{aligned}$$