Mathematics 250: Lecture 4

Parametrized Curves

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Parametrized curves

- If I is an interval in $\mathbb{R}$, we call a continuous function $F : I \to \mathbb{R}^q$ a *parametrized curve*.
- We call the image C in $\mathbb{R}^q$ of a parametrized curve F a *curve*.
- That is, if $F : \mathbb{R} \to \mathbb{R}^q$ is a parametrized curve, then

$$C = \{Y \in \mathbb{R}^q : Y = F(x) \text{ for some } x \in I\}$$

  is a curve, and F is a *parametrization* of C.
- If we write $F(t) = (f_1(t), f_2(t), \ldots, f_q(t))$, then we call

$$\begin{aligned} x_1 &= f_1(t) \\ x_2 &= f_2(t) \\ \vdots &= \vdots \\ x_q &= f_q(t) \end{aligned}$$

  parametric equations of C.

Example: lines

- Given vectors Y_0 and Z in $\mathbb{R}^q$,

$$F(t) = Y_0 + tZ, \quad -\infty < t < \infty,$$

  parametrizes a line in $\mathbb{R}^q$.
- For example,

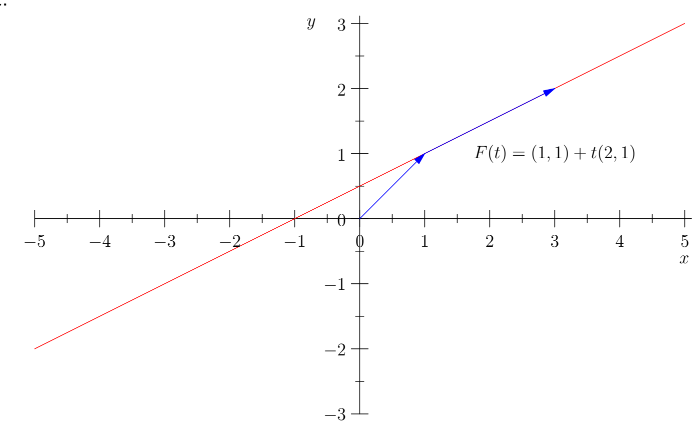
$$F(t) = (1, 1) + t(2, 1)$$

  parametrizes the line L passing through the points $(1, 1)$ and $(3, 2)$ in $\mathbb{R}^2$.
- The parametric equations for this parametrization are

$$\begin{aligned} x &= 1 + 2t \\ y &= 1 + t. \end{aligned}$$

Example (cont'd)

- Plot of L:



Example (cont'd)

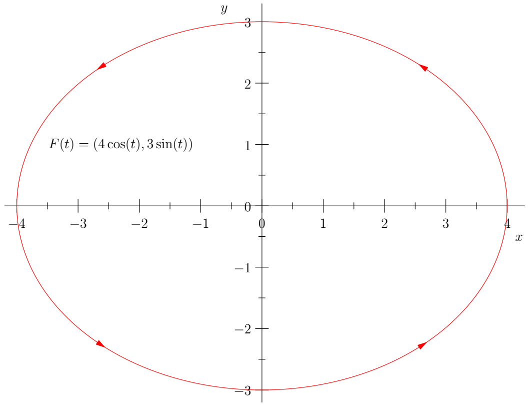
- Note: Both
$$G(t) = (1,1) + t^3(2,1)$$
and
$$H(t) = (3,2) - t(2,1)$$
parametrize the same line L, but give different motions on the line.
- Note: If we restrict the interval for t, then we would have a parametrization of a part of the line L.
- For example,
$$F(t) = (1,1) + t(2,1), \quad 0 \le t \le 1,$$
parametrizes the line segment from $(1,1)$ to $(3,2)$.

Example: ellipses

- Let $a > 0$, $b > 0$, and define
$$F(t) = (a\cos(t), b\sin(t)), \quad 0 \le t < 2\pi.$$
- Let C be the curve parametrized by F.
- The parametric equations of C are
$$x = a\cos(t)$$
$$y = b\sin(t).$$
- Note: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \cos^2(t) + \sin^2(t) = 1.$
- That is, C is an ellipse, with center at the origin and passing through the points $(a,0)$, $(0,b)$, $(-a,0)$, and $(0,-b)$.
- Note: This parametrization traverses C exactly once in the counterclockwise direction.
- Note: Allowing $0 \le t \le 4\pi$ would traverse the curve twice.

Example (cont'd)

- Ellipse parametrized by $F(t) = (4\cos(t), 3\sin(t))$:

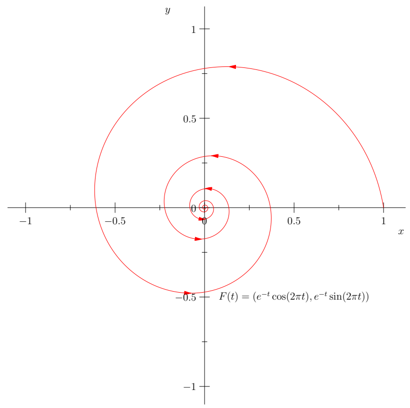


Spiral

- Let $F(t) = (e^{-t}\cos(2\pi t), e^{-t}\sin(2\pi t))$, $0 \le t < \infty$.
- Note:
  - $\|F(t)\| = e^{-t}\|(\cos(2\pi t), \sin(2\pi t))\| = e^{-t}$.
  - $\lim_{t\to\infty} F(t) = (0,0)$.
- $F(t)$ parametrizes a curve which spirals in towards the origin.

Example (cont'd)

- Spiral parametrized by $F(t) = (e^{-t}\cos(2\pi t), e^{-t}\sin(2\pi t))$:

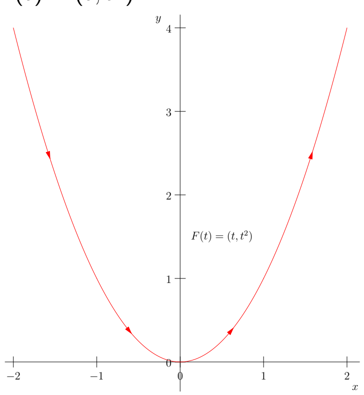


Example: graphs

- Suppose C is the graph of a function $f : \mathbb{R} \to \mathbb{R}$.
- Then $F(t) = (t, f(t))$ is a parametrization of C.
- For example, $F(t) = (t, t^2)$, $-\infty < t < \infty$, is a parametrization of the parabola $y = x^2$.

Example (cont'd)

- Parabola parametrized by $F(t) = (t, t^2)$:

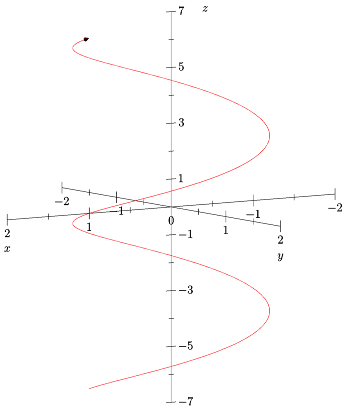


Example: helix

- $F(t) = (\cos(t), \sin(t), t)$, $-\infty < t < \infty$, parametrizes a helix about the z-axis.

Example (cont'd)

- Helix parametrized by $F(t) = (\cos(t), \sin(t), t)$, $-2\pi \le t \le 2\pi$:



Using wxMaxima

- To draw a two-dimensional parametric plot with wxMaxima:
  - Select `Plot2d` from the `Plot` menu.
  - Click on `Special` and select `Parametric plot`.
- To draw a three-dimensional parametric plot with wxMaxima:
  - Type the command `load(draw)`.
  - Note: When typing commands in wxMaxima, use `Shift+Enter` to execute the command.
  - Then, for example, the command
    `wxdraw3d(nticks=300,[parametric(cos(t),sin(t),t,t,-2*%pi,2*%pi)]);`
    will plot a helix.
  - Alternatively:
    `draw3d(nticks=300,[parametric(cos(t),sin(t),t,t,-2*%pi,2*%pi)]);`
    will draw the helix in a separate window.