Mathematics 250: Lecture 6

Derivatives and Acceleration

Dan Sloughter

Furman University

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Properties of derivatives

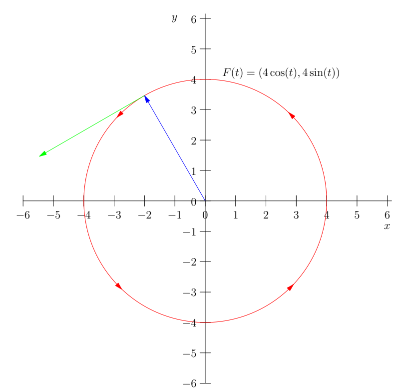
- Suppose $F : \mathbb{R} \to \mathbb{R}^q$, $G : \mathbb{R} \to \mathbb{R}^q$, and $h : \mathbb{R} \to \mathbb{R}$ are all differentiable and α is a scalar
- Then
  - $\frac{d}{dt}(F(t) + G(t)) = \frac{dF}{dt}(t) + \frac{dG}{dt}(t)$,
  - $\frac{d}{dt}\alpha F(t) = \alpha \frac{dF}{dt}(t)$,
  - $\frac{d}{dt}h(t)F(t) = h'(t)F(t) + h(t)F'(t)$,
  - $\frac{d}{dt}(F(t) \cdot G(t)) = F(t) \cdot G'(t) + F'(t) \cdot G(t)$.
- Chain rule: If $F : \mathbb{R} \to \mathbb{R}^q$ and $h : \mathbb{R} \to \mathbb{R}$ are both differentiable, then $F \circ h$ is differentiable and
$$\frac{d}{dt}F(h(t)) = F'(h(t))h'(t).$$

Example

- Suppose $F : \mathbb{R} \to \mathbb{R}^q$ and $\|F(t)\|$ is a constant for all t.
- For example, we could have $F(t) = (a\cos(t), a\sin(t))$, where $a > 0$.
- If $\varphi(t) = \|F(t)\|^2$, then $\varphi'(t) = 0$ for all t.
- So we have
$$0 = \frac{d}{dt}\|F(t)\|^2 = \frac{d}{dt}(F(t) \cdot F(t)) = 2F(t) \cdot F'(t).$$
- Hence $F(t) \perp F'(t)$ for all t.
- That is, the position vector is orthogonal to the tangent vector.

Example (cont'd)

- Plot of $F(t) = (4\cos(t), 4\sin(t))$, with position and tangent vectors at $t = \frac{2\pi}{3}$:



Acceleration

- If $F : \mathbb{R} \to \mathbb{R}^q$ is differentiable, and its derivative F' is also differentiable, then we call

$$F''(t) = (f_1''(t), f_2''(t), \ldots, f_q''(t))$$

the *second derivative* of F.

- Note: If $F(t)$ gives the position of a particle at time t, then $F''(t)$ is the *acceleration* of the particle at time t.

Example

- Suppose $F(t) = (\cos(t), \sin(t), t)$ gives the position of an object moving along a helix in $\mathbb{R}^3$ at time t.
- Then the velocity of the particle at time t is

$$F'(t) = (-\sin(t), \cos(t), 1).$$

- And the acceleration is

$$F''(t) = (-\cos(t), -\sin(t), 0).$$

- For example, at time $t = \pi$, the particle is at $F(\pi) = (-1, 0, \pi)$, with velocity

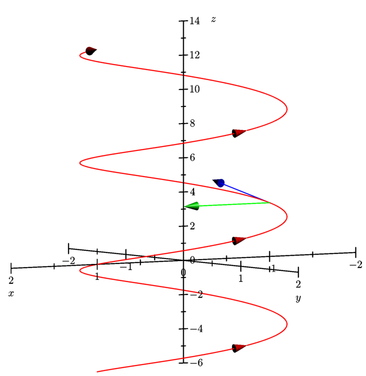
$$F'(\pi) = (0, -1, 1),$$

and acceleration

$$F''(\pi) = (1, 0, 0).$$

Example (cont'd)

- Plot of $F(t) = (\cos(t), \sin(t), t)$ with velocity and acceleration vectors at time $t = \pi$:



Example: projectile motion

- Suppose an object of mass m is projected into the air from the surface of the earth with an initial speed of s_0 at an angle θ with the horizontal.
- Let $X(t)$ be the position of the object at time t, with $X(0) = (0, 0)$.
- Note: Ignoring air resistance, the only force acting on the object is the force of gravity.
- Then $m\ddot{X}(t) = (0, -mg)$.
- And so the velocity of the object must be, for some constant vector V_0,

$$\dot{X}(t) = t(0, -g) + V_0$$

- But then $s_0(\cos(\theta), \sin(\theta)) = \dot{X}(0) = V_0$, so

$$\dot{X}(t) = t(0, -g) + (s_0 \cos(\theta), s_0 \sin(\theta)).$$

Example (cont'd)

- It now follows that, for some constant vector X_0,

$$X(t) = \frac{1}{2}t^2(0, -g) + t(s_0\cos(\theta), s_0\sin(\theta)) + X_0.$$

- Then $(0,0) = X(0) = X_0$.
- Thus

$$X(t) = -\frac{1}{2}gt^2(0,1) + (s_0\cos(\theta), s_0\sin(\theta))t.$$

- That is, the parametric equations for the path of the object

$$x(t) = s_0\cos(\theta)t,$$
$$y(t) = -\frac{1}{2}gt^2 + s_0\sin(\theta)t.$$

- Note: If the first equation is solved for t, and then substituted into the second, we see that the path of the projectile is a parabola.

Example (cont'd)

- Also, note that that the object strikes the ground when $y(t) = 0$, that is when

$$t = \frac{2s_0\sin(\theta)}{g}.$$

- At that time, the horizontal distance traveled is

$$x\left(\frac{2s_0\sin(\theta)}{g}\right) = \frac{2s_0^2\sin(\theta)\cos(\theta)}{g} = \frac{s_0^2}{g}\sin(2\theta).$$

- Note: The object will travel a maximal distance when $\sin(2\theta) = 1$, that is, when $\theta = \frac{\pi}{4}$.

Example (cont'd)

- Projectile motion with initial velocity $(10\sqrt{2}, 10\sqrt{2})$, that is, with $s_0 = 20$ and $\theta = \frac{\pi}{4}$: