Mathematics 255: Lecture 11

Applications: Growth and Decay

Dan Sloughter

Furman University

February 3, 2017

Exponential growth and decay

- Suppose y is a quantity whose rate of growth is proportional to itself.
- That suppose, for some constant a,
$$\frac{dy}{dt} = ay.$$
- If $y(0) = y_0$, we have
$$y = y_0 e^{at}.$$
- Note:
  - If $a > 0$, this is *exponential growth*.
  - If $a < 0$, this is *exponential decay*.

Radioactive decay

- Suppose y is the amount of a radioactive element present at time t.
- Then, for some positive constant k,
$$\frac{dy}{dt} = -ky.$$
- Note: This is also called a *first-order reaction*.
- If the initial amount is y_0, then
$$y = y_0 e^{-kt}.$$

Example

- Carbon-14 has a half-life of 5730 years.
- That is, given an initial amount y_0, we have
$$\frac{1}{2}y_0 = y_0 e^{-5730k}$$
for some positive constant k.
- Solving for k, we have
$$k = \frac{\log(2)}{5730}.$$

Example (cont'd)

- A piece of charcoal from a tree destroyed by the volcanic eruption which formed Crater Lake was found to have 44.5% of it's original carbon-14.
- If T is the number of years since the eruption, then, if y_0 is the original amount of carbon-14,
$$0.445 y_0 = y_0 e^{-kT}$$
- Solving for T,
$$T = -\frac{\log(0.445)}{k} = -\frac{5730 \log(0.445)}{\log(2)} \approx 6693 \text{ years.}$$

Population growth

- Suppose $N(t)$ is the size of a certain population at time t and $N_0 = N(t_0)$.
- Natural, or uninhibited, growth model:
  - For some positive constant K, $\frac{dN}{dt} = KN$.
  - Hence $N = N_0 e^{K(t-t_0)}$.
- Logistic, or inhibited, growth model:
  - For some positive constants K and C, $\frac{dN}{dt} = KN(C - N)$.
  - Equilibrium solutions: $N \equiv 0$ and $N \equiv C$.
  - For $0 < N_0 < C$,
$$N = \frac{C}{1 + \left(\frac{C}{N_0} - 1\right) e^{-KC(t-t_0)}}.$$
  - Limiting population: $\lim_{t \to \infty} N = C$.

Newton's law of cooling

- Suppose T is the temperature of an object at time t.
- Suppose the temperature of the surrounding environment is T_s.
- Newton's law of cooling: For some positive constant k,
$$\frac{dT}{dt} = -k(T - T_s).$$
- Hence T satisfies a first-order linear equation:
$$\frac{dT}{dt} + kT = kT_s.$$
- So if $T(0) = T_0$, we have
$$Te^{kt} - T_0 = T_s e^{kt} - T_s.$$
- Thus
$$T = T_s + (T_0 - T_s)e^{-kt}.$$

Example

- Suppose a cup of coffee, initially at a temperature of 210° degrees, is placed in a room held at a constant 72°.
- Suppose after 10 minutes the coffee has cooled to 150°.
- Q: What is the temperature of the coffee after 20 minutes?
- If T is the temperature of the coffee after t minutes, then
$$T = 72 + 138e^{-kt}$$
for some constant k.
- At $t = 10$, we have
$$150 = 72 + 138e^{-10k}.$$
- So $k = -\frac{\log\left(\frac{78}{138}\right)}{10} \approx 0.0570$.

Example (cont'd)

- Hence, after 20 minutes, the temperature of the coffee is

$$T(20) = 72 + 138e^{-20k} \approx 116^\circ.$$