Mathematics 255: Lecture 12

Solutions and Picard Iteration

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Picard's theorem

- Suppose $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are both continuous on a rectangle
$$R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\},$$
where $a < b$ and $c < d$ are constants.
- Suppose $a < x_0 < b$ and $c < y_0 < d$.
- Then the initial-value problem
$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$
has a unique solution $y(x)$ defined on an open interval $(x_0 - \alpha, x_0 + \alpha)$ for some $\alpha > 0$.
- Note:
  - Continuity of f is enough to guarantee the existence of a solution.
  - Continuity of $\frac{\partial f}{\partial y}$ guarantees the uniqueness.

Example

- Both $y(x) \equiv 0$ and
$$y(x) = \frac{1}{27}x^3$$
are solutions of the initial-value problem
$$\frac{dy}{dx} = y^{\frac{2}{3}}, \quad y(0) = 0.$$
- Problem: $f(x, y) = y^{\frac{2}{3}}$, so
$$\frac{\partial f}{\partial y}(x, y) = \frac{2}{3y^{\frac{1}{3}}},$$
which is not continuous on any rectangle containing $(0, 0)$.

Integral equations

- Note: The differential equation
$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$
is equivalent to the *integral equation*
$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

Picard iterates

- To approximate a solution to the differential equation
$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0,$$
or, equivalently, the integral equation
$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt,$$
let $y_0(x) \equiv y_0$ and then compute, iteratively,
$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt.$$
- The functions y_n, the *Picard iterates*, are, under certain assumptions, successive approximations to the solution.

Example

- Consider
$$\frac{dy}{dx} = y, \quad y(0) = 1.$$
- Let $y_0(x) = 1$ and $f(x,y) = y$. Then
  - $y_1(x) = y_0 + \int_0^x f(t, y_0(t))dt = 1 + \int_0^x dt = 1 + x$
  - $y_2(x) = 1 + \int_0^x (1+t)dt = 1 + x + \frac{1}{2}x^2$
  - $y_3(x) = 1 + \int_0^x \left(1 + t + \frac{1}{2}t^2\right) dt = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3.$

Example (cont'd)

- After n iterations, we have
$$y_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$
- Note:
$$\lim_{n\to\infty} y_n(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = e^x,$$
which we know from our earlier work to be the solution to the equation.

Example

- Consider
$$\frac{dy}{dx} = 2xy, \quad y(0) = 1.$$
- Let $y_0(x) = 1$ and $f(x,y) = 2xy$. Then
  - $y_1(x) = y_0 + \int_0^x f(t, y_0(t))dt = 1 + \int_0^x 2tdt = 1 + x^2.$
  - $y_2(x) = 1 + \int_0^x 2t(1+t^2)dt = 1 + x^2 + \frac{1}{2}x^4.$
  - $y_3(x) = 1 + \int_0^x 2t\left(1 + t^2 + \frac{1}{2}t^4\right) dt = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{3!}x^6.$
  - $y_4(x) = 1 + \int_0^x 2t\left(1 + t^2 + \frac{1}{2}t^4 + \frac{1}{3!}t^6\right) dt = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8.$

Example (cont'd)

- After n iterations, we have

$$y_n(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{3!}x^6 + \cdots + \frac{1}{n!}x^{2n}.$$

- Note:

$$\lim_{n\to\infty} y_n(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^{2n} = e^{x^2},$$

  which we know from our earlier work to be the solution to the equation.

Linear equations

- Suppose, for $i = 1, 2, \ldots, n$, $a_i(x)$ and $f(x)$ are continuous on some open interval I.
- Suppose $a_0(x) \neq 0$ for all $x \in I$.
- Suppose $x_0 \in I$.
- Then, for any real numbers b_0, b_1, $\ldots$, b_{n-1}, the differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x)$$

  has a unique solution on I satisfying the initial conditions

$$y(x_0) = b_0, y'(x_0) = b_1, \ldots, y^{(n-1)}(x_0) = b_{n-1}.$$