Mathematics 255: Lecture 19

Euler's Equation

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Euler's equation

- If p and q are constants and $f(x)$ is a continuous function, *Euler's equation* is a differential equation of the form
$$x^2y'' + pxy' + qy = f(x).$$
- Note:
  - The existence and uniqueness theorem applies to this equation only for intervals which do not contain 0.
  - So we will consider the equation only for $x > 0$.

The homogeneous equation

- Consider the homogeneous equation
$$x^2y'' + pxy' + qy = 0.$$
- Let m be a constant and let $y = x^m$.
- Then $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$.
- Hence y is a solution if and only if
$$0 = m(m-1)x^m + pmx^m + qx^m = (m(m-1) + pm + q)x^m = (m^2 + (p-1)m + q)x^m.$$
- Hence y is a solution if and only if the *index* m satisfies the *indicial equation*
$$m^2 + (p-1)m + q = 0.$$

Distinct real roots

- Suppose the indicial equation has distinct real roots m_1 and m_2.
- Then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ are solutions.
- Moreover,
$$W[y_1(x), y_2(x)] = \begin{vmatrix} x^{m_1} & x^{m_2} \\ m_1x^{m_1-1} & m_2x^{m_2-1} \end{vmatrix} = (m_2 - m_1)x^{m_1+m_2-1}.$$
- Hence y_1 and y_2 are linearly independent.
- And so the general solution is
$$y = c_1x^{m_1} + c_2x^{m_2}.$$

Example

- Consider the equation
$$x^2y'' + xy' - y = 0.$$
- The indicial equation is
$$0 = m^2 - 1 = (m-1)(m+1).$$
- So the general solution is
$$y = c_1x + \frac{c_2}{x}.$$

Double roots

- Suppose
$$0 = m^2 + (p-1)m + q$$
has a double root
$$m = \frac{1-p}{2}.$$
- Then $y_1 = x^m$ is one solution.
- Using variation of parameters, we look for a second solution of the form $y = u(x)x^m$.
- Then
$$y' = mux^{m-1} + u'x^m \text{ and } y'' = m(m-1)ux^{m-2} + 2mu'x^{m-1} + u''x^m.$$

Double roots (cont'd)

- And so we want
$$\begin{aligned} 0 &= m(m-1)ux^m + 2mu'x^{m+1} + u''x^{m+2} + pmux^m + pu'x^{m+1} + qux^m \\ &= (m(m-1) + pm + q)ux^m + (2m+p)u'x^{m+1} + u''x^{m+2} \\ &= x^{m+1}(u' + xu''). \end{aligned}$$
- Hence we need $xu'' + u' = 0$.
- Letting $v = u'$, this becomes the first-order linear equation
$$v' + \frac{1}{x}v = 0.$$

Double roots (cont'd)

- Hence
$$v = k_1e^{-\int \frac{1}{x}\,dx} = \frac{k_1}{x}.$$
- And so $u = k_1 \log(x) + k_2$.
- Hence we may take $u = \log(x)$.
- And so a second solution is $y_2 = x^m \log(x)$
- y_1 and y_2 are linearly independent, so the general solution is
$$y = c_1x^m + c_2x^m \log(x).$$

Example

- Consider the equation
$$x^2y'' + 3xy' + y = 0.$$
- Then the indicial equation is
$$0 = m^2 + 2m + 1 = (m+1)^2.$$
- So the general solution is
$$y = \frac{c_1}{x} + \frac{c_2 \log(x)}{x}.$$

Complex roots

- Suppose
$$0 = m^2 + (p-1)m + q$$
has complex roots $m_{1,2} = a \pm ib$.
- Then one complex solution is
$$z = x^{a+ib} = x^a x^{ib} = x^a e^{ib\log(x)} = x^a(\cos(b\log(x)) + i\sin(b\log(x))).$$
- Hence
$$y_1 = x^a\cos(b\log(x)) \text{ and } y_2 = x^a\sin(b\log(x))).$$
- Thus the general solution is
$$y = x^a(c_1\cos(b\log(x)) + c_2\sin(b\log(x))).$$

Example

- Consider the equation
$$x^2y'' + 2xy' + y = 0.$$
- Then the indicial equation is
$$0 = m^2 + m + 1.$$
- Then $m = \frac{-1\pm\sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.
- And so the general solution is
$$y = \frac{1}{\sqrt{x}}\left(c_1\cos\left(\frac{\sqrt{3}}{2}\log(x)\right) + c_2\sin\left(\frac{\sqrt{3}}{2}\log(x)\right)\right).$$