Mathematics 255: Lecture 2

Motion

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Two laws from physics

- Newton's second law of motion:
  - Suppose an object of mass m moves along a straight line, with position $x(t)$ at time t.
  - The velocity of the object is
$$v = \frac{dx}{dt} = \dot{x}.$$
  - The acceleration of the object is
$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x}.$$
  - If F is the force acting on the object, then
$$F = ma = m\ddot{x}.$$
  - Note: F may be a function of any, or all, of t, x, and $\dot{x}$.

Two laws of physics

- Newton's universal law of gravitation:
  - If two objects of masses m and M are a distance x apart, the gravitational force F between the two objects is
$$F = \frac{GmM}{x^2},$$
where G is the gravitational constant.
  - $G \approx 6.67 \times 10^{-11} \text{m}^2/\text{kg s}^2$
- Note: At the surface of the earth, the force of the earth's gravity on an object of mass m is mg, where $g \approx 9.8$ meters per second per second.

Body in free fall

- Suppose an object of mass m falls freely near the surface of the earth with no air resistance.
  - Let x denote the position of the object along an axis oriented positively downward.
  - Then $mg = m\ddot{x}$, or $g = \ddot{x}$.
- Now suppose air resistance adds a force which opposes the motion of the object.
  - If the force of air resistance is proportional to $\dot{x}^n$ for some n, then
$$mg - r\dot{x}^n = m\ddot{x},$$
for some constant r.

Mass-spring system

- Consider an object of mass m attached to the end of spring.
- Let s be the amount the mass stretches the spring to reach the *equilibrium position*.
- Let x be the position of the object at time t, measured from the equilibrium position, with the positive direction downward.
- By Hooke's law, we have

$$ks = mg$$

  for some constant $k > 0$, the *spring constant*.

Mass-spring system (cont'd)

- We assume the only forces acting on the spring are:
  - mg, the force of gravity,
  - $-k(x+s)$, the restoring force of the spring,
  - $-r\dot{x}$, $r \geq 0$, a damping force due to the resistance of the surrounding medium,
  - $F(t)$, an external driving force, assumed to be a function of time alone.
- By Newton's second law, we then have

$$m\ddot{x} = mg - k(x+s) - r\dot{x} + F(t),$$

- Recalling that $mg = ks$, this becomes

$$m\ddot{x} + r\dot{x} + kx = F(t).$$

Motion in space

- Suppose particle of mass m has position $\vec{r} = (x, y, z)$ at time t.
- The *velocity vector* of the particle is

$$\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z}).$$

- The *acceleration vector* of the particle is

$$\ddot{\vec{r}} = (\ddot{x}, \ddot{y}, \ddot{z}).$$

- If the particle is subject to a force $\vec{F} = (F_1, F_2, F_3)$, then, by Newton's second law,

$$\vec{F} = m\ddot{\vec{r}}.$$

Motion in space (cont'd)

- The latter vector equation is equivalent to three scalar equations:

$$F_1 = m\ddot{x}$$
$$F_2 = m\ddot{y}$$
$$F_3 = m\ddot{z}$$

- In general, these equations may not be solved separately since each of F_1, F_2, and F_3 may depend on all of x, y, z, and t.

Constrained motion

- Suppose a particle of mass m is constrained to move along a fixed curve C in $\mathbb{R}^3$.
- Let $\vec{r}(t) = (x(t), y(t), z(t))$ be the position of the particle at time t.
- Note:
  - $\dot{\vec{r}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$ is a vector which is tangent to C at the point $r(\vec{t})$.
  - The *speed* of the particle at time t is

$$\|\dot{\vec{r}}(t)\| = \sqrt{(\dot{x}(t))^2 + \dot{y}(t))^2 + \dot{z}(t))^2}.$$

  - The length of the curve traveled from time 0 to time t is

$$s(t) = \int_0^t \|\dot{\vec{r}}(u)\| du.$$

Constrained motion (cont'd)

- It now follows that

$$\dot{s} = \|\dot{\vec{r}}\| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}.$$

- And

$$\ddot{s} = \frac{1}{2\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}(2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} + 2\dot{z}\ddot{z}) = \frac{\dot{\vec{r}} \cdot \ddot{\vec{r}}}{\|\dot{\vec{r}}\|}.$$

- Now suppose F is the the force acting on the particle
- Let $f(s,t)$ be the component of the force acting on the particle at a distance s along the curve and at time t in the direction of $\dot{\vec{r}}(t)$.
- Then

$$f(s,t) = \frac{F \cdot \dot{\vec{r}}}{\|\dot{\vec{r}}\|}.$$

Constrained (cont'd)

- Now $F = m\ddot{\vec{r}}$, so

$$f(s,t) = \frac{m\ddot{\vec{r}} \cdot \dot{\vec{r}}}{\|\dot{\vec{r}}\|} = m\ddot{s}.$$

- Example: motion of a pendulum (Problem 3 in Problems 1-3).

General and particular

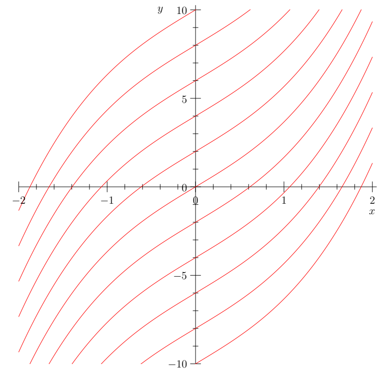
- We call a single solution of a differential equation a *particular solution*.
- We call the set of all solutions of a differential equation the *general solution*.
- We call the curve defined by a particular solution an *integral curve*.

Example

- Consider the equation
$$\frac{dy}{dx} = 2x^2 + 3.$$
- Then $y = \frac{2}{3}x^3 + 3x$ is a particular solution.
- The set of all functions of the form
$$y = \frac{2}{3}x^3 + 3x + c,$$
where c can be any real number, is the general solution.

Example (cont'd)

- Some integral curves of $\frac{dy}{dx} = 2x^2 + 3$:



Implicit solutions

- Consider the equation
$$y\frac{dy}{dx} + x = 0.$$
- If $c > 0$ is a real number and y and x satisfy
$$y^2 + x^2 = c.$$
then y is a solution of the equation.
- Check: using implicit differentiation,
$$2y\frac{dy}{dx} + 2x = 0, \text{ so } y\frac{dy}{dx} + x = 0.$$
- $y^2 + x^2 = c$ *implicitly* defines solutions of the equation.
- We may solve for y to obtain *explicit* solutions:
$$y = \sqrt{c - x^2} \text{ or } y = -\sqrt{c - x^2}.$$
- Note: these solutions are defined only on the interval $(-\sqrt{c}, \sqrt{c})$.