Mathematics 255: Lecture 5

First-order Linear Equations

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Definitions

- Recall: We call a differential equation of the form
$$a(x)\frac{dy}{dx} + b(x)y = c(x)$$
a first-order linear differential equation.
- On an interval where $a(x) \neq 0$, we may let $P(x) = \frac{b(x)}{a(x)}$ and $Q(x) = \frac{c(x)}{a(x)}$, and then write the equation as
$$\frac{dy}{dx} + P(x)y = Q(x).$$
- Note: Unless otherwise noted, we assume $P(x)$ and $Q(x)$ are continuous functions.
- Recall: If $Q(x) \equiv 0$, we say the equation is *homogeneous*; otherwise, it is *nonhomogeneous*.

Homogeneous equations

- Consider the first-order, linear, homogeneous equation
$$\frac{dy}{dt} + P(x)y = 0.$$
- One solution: $y \equiv 0$.
- Working on an interval where $y(x) \neq 0$, the equation is separable, and we have, for an arbitrary constant c,
$$\int P(x)dx + \int \frac{1}{y}dy = c_1$$
- Hence
$$\log(|y|) = c_1 - \int P(x)dx.$$
- And so
$$y = \pm e^{c_1} e^{-\int P(x)dx}.$$

Homogeneous equations (cont'd)

- It follows that the general solution is, for an arbitrary constant c,
$$y = ce^{-\int P(x)dx}.$$

Example

- Consider the equation
$$\frac{dy}{dx} - xy = 0.$$
- In our notation, $P(x) = -x$, so the general solution is
$$y = ce^{\int x dx} = ce^{\frac{x^2}{2}}.$$

Initial-value problems

- Now consider the first-order linear equation with an initial condition:
$$\frac{dy}{dx} + P(x)y = 0, \quad y(x_0) = y_0.$$
- Using a definite integral, we have
$$\int_{x_0}^{x} P(t)dt + \int_{y_0}^{y} \frac{1}{t}dt = 0.$$
- Hence
$$\log(|y|) - \log(|y_0|) = -\int_{x_0}^{x} P(t)dt).$$
- And so
$$\log\left|\frac{y}{y_0}\right| = -\int_{x_0}^{x} P(t)dt.$$
- Solving for y, we find $y = y_0 e^{-\int_{x_0}^{x} P(t)dt}$.

Example

- Consider the initial-value problem
$$\frac{dy}{dx} + 4x^2 y = 0, \quad y(1) = 5.$$
- Here $P(x) = 4x^2$, and so
$$\int_{1}^{x} P(t)dt = \int_{1}^{x} 4t^2 dt = \frac{4}{3}(x^3 - 1).$$
- Hence
$$y = 5e^{\frac{4}{3}(1-x^3)}.$$

Nonhomogeneous equations

- Now consider the nonhomogeneous equation
$$\frac{dy}{dx} + P(x)y = Q(x).$$
- Note:
  - Suppose we can find a function $\mathrm{I}(x)$ for which
$$\frac{d}{dx}\mathrm{I}(x)y(x) = \mathrm{I}(x)\frac{dy}{dx} + \mathrm{I}(x)P(x)y.$$
  - Then it would follow that
$$\mathrm{I}(x)y(x) = \int \mathrm{I}(x)Q(x)dx.$$
- Such a function I must, by the product rule, satisfy
$$\frac{d\mathrm{I}}{dx} = P(x)\mathrm{I}.$$

Integrating factors

- Note: The last equation is first-order homogeneous linear equation.
- So
$$\mathrm{I}(x) = e^{\int P(x)dx}.$$
- We call $\mathrm{I}(x)$ an *integrating factor*.

Example

- Consider the equation
$$\frac{dy}{dx} + 2xy = 3x.$$
- Then $P(x) = 2x$, so an integrating factor is
$$\mathrm{I}(x) = e^{\int 2xdx} = e^{x^2}.$$
- Then
$$e^{x^2}\left(\frac{dy}{dx} + 2xy\right) = 3xe^{x^2}.$$
- Or
$$\frac{d}{dt}e^{x^2}y = 3xe^{x^2}.$$

Example (cont'd)

- Hence, for an arbitrary constant c,
$$ye^{x^2} = \int 3xe^{x^2}dx = \frac{3}{2}e^{x^2} + c.$$
- Thus
$$y = \frac{3}{2} + ce^{-x^2}.$$

Example

- Now consider the initial-value problem
$$\frac{dy}{dx} - y = \sin(x), \quad y(0) = 5.$$
- An integrating factor is
$$\mathrm{I}(x) = e^{-\int dx} = e^{-x}.$$
- And so
$$\frac{d}{dx}ye^{-x} = e^{-x}\sin(x).$$
- Integrating both sides,
$$y(t)e^{-t}\Big|_0^x = \int_0^x e^{-t}\sin(t)dt.$$

Example (cont'd)

- Since
$$\int_0^x e^{-t}\sin(t)dt = \frac{1}{2} - \frac{e^{-x}}{2}\left(\sin(x)+\cos(x)\right),$$
we have
$$ye^{-x} - 5 = \frac{1}{2} - \frac{e^{-x}}{2}\left(\sin(x)+\cos(x)\right).$$
- Thus
$$y = \frac{11}{2}e^{x} - \frac{1}{2}\left(\sin(x)+\cos(x)\right).$$