Mathematics 340: Lecture 14

Expectation

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Expected value

- Definition:
  - Suppose X is a discrete random variable with range A and probability mass function p_X.
  - Suppose the series
    $$\sum_{x \in A} |x| p_X(x)$$
    converges.
  - We call
    $$\mathrm{E}(X) = \sum_{x \in A} x p_X(x)$$
    the *expected value*, or *mean*, of X.
- Note:
  - $\mathrm{E}(X)$ is a weighted average of the possible values of X.
  - Physically, $\mathrm{E}(X)$ is the center of mass of the probability mass function.

Example

- Suppose two fair dice are rolled.
- Let X be the sum of the two dice.
- Then
  $$\begin{aligned}\mathrm{E}(X) &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{9} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{1}{6} \\ &\quad + 8 \cdot \frac{5}{36} + 9 \cdot \frac{1}{9} + 10 \cdot \frac{1}{12} + 11 \cdot \frac{1}{18} + 12 \cdot \frac{1}{36} \\ &= 7.\end{aligned}$$
- Interpretation
  - Suppose you play a game in which you win one dollar times the sum of the rolls of two dice.
  - Then 7 would be a fair entrance fee to play this game.

Example: St. Petersburg paradox

- Suppose a fair coin is tossed three times and you win 2^n dollars if the first head is on the nth toss (and nothing if all three tosses are tails).
- Let X be your winnings.
- Then
  $$\mathrm{E}(X) = 0 \cdot \frac{1}{8} + 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} = 3.$$
- Now suppose we do not put a limit on the number of tosses. That is, the coin is tossed until the first head, and you win 2^n dollars if the first head is on the nth toss.
- Then, if X is the amount you win,
  $$\sum_{n=1}^{\infty} 2^n P(X = 2^n) = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} 1,$$
  which does not converge.
- How much would you be willing to pay to play this game?

Example

- Suppose $X \sim \text{Bern}(p)$, with probability mass function p_X.
- Then
$$E(X) = 0 \cdot p_X(0) + 1 \cdot p_X(1) = 0 \cdot (1-p) + 1 \cdot p = p.$$

Constant multiples

- Suppose X is a random variable with probability mass function p_X and support A.
- Let $c \neq 0$ be a constant and let $Y = cX$.
- Let B be the support of Y and let p_Y be the probability mass function of Y.
- Then
$$\begin{aligned} E(Y) &= \sum_{y \in B} y p_Y(y) = \sum_{y \in B} y P(Y = y) \\ &= \sum_{x \in A} cx P(Y = cx) = \sum_{x \in A} cx P(X = x) \\ &= c \sum_{x \in A} x p_X(x) \\ &= c\, E(X). \end{aligned}$$
- That is $E(cX) = c\, E(X)$.

Sums

- Suppose X and Y are random variables with probability mass functions p_X and p_Y.
- Let A and B be the supports of X and Y, respectively.
- Let $Z = X + Y$.
- Let p_Z be the probability mass function and C the support of Z.

Sums (cont'd)

- Then
$$\begin{aligned} E(Z) &= \sum_{z \in C} z p_Z(z) = \sum_{z \in C} z P(X + Y = z) = \sum_{z \in C} z \sum\sum_{\substack{x \in A, y \in B \\ x+y=z}} P(X = x, Y = y) \\ &= \sum_{x \in A} \sum_{y \in B} (x+y) P(X = x, Y = y) \\ &= \sum_{x \in A} \sum_{y \in B} x P(X = x, Y = y) + \sum_{x \in A} \sum_{y \in B} y P(X = x, Y = y) \\ &= \sum_{x \in A} x \sum_{y \in B} P(X = x, Y = y) + \sum_{y \in B} y \sum_{x \in A} P(X = x, Y = y) \\ &= \sum_{x \in A} x p_X(x) + \sum_{y \in B} y p_Y(y) \\ &= E(X) + E(Y). \end{aligned}$$
- That is, $E(X + Y) = E(X) + E(Y)$.

Example

- Suppose two fair dice are rolled.
- Let X be the result of the first roll and Y the result of the second roll.
- Then
$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2}.$$
- So we also have $E(Y) = \frac{7}{2}$.
- If $Z = X + Y$, then
$$E(Z) = E(X) + E(Y) = \frac{7}{2} + \frac{7}{2} = 7,$$
as we saw in an earlier example.

Example: binomial random variables

- Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with probability of success p.
- Let $Y = X_1 + X_2 + \cdots + X_n$.
- Then $Y \sim \text{Bin}(n, p)$.
- And
$$E(Y) = E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n) = np.$$

Example: hypergeometric random variables

- Suppose a sample of n chips is drawn, without replacement, from an urn with w white chips and b black chips.
- Let Y be the number of white chips in the sample.
- Then $Y \sim \text{HGeom}(w, b, n)$.
- Define Bernoulli random variables, $X_1, X_2, \ldots, X_n$, such that $X_i = 1$ if the ith chip drawn is white and 0 otherwise, for $i = 1, 2, \ldots, n$.
- Then $Y = X_1 + X_2 + \cdots + X_n$.
- Note: $E(X_i) = \frac{w}{w+b}$, $i = 1, 2, \ldots, n$.
- Thus
$$E(Y) = E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n) = \frac{nw}{w+b}.$$