Mathematics 340: Lecture 16

Variance

Dan Sloughter

Furman University

October 8, 2015

Variance

- If X is a random variable with mean μ, then we call

$$\text{Var}(X) = \text{E}((X - \mu)^2)$$

the *variance* of X.

- Notation: we often use σ^2, or σ_X^2 to denote $\text{Var}(X)$.
- We call $\text{SD}(X) = \sqrt{\text{Var}(X)}$, or simply σ or σ_X, the *standard deviation* of X.

Computation

- If X is a random variable with mean μ, then

$$\text{Var}(X) = \text{E}(X^2) - \mu^2.$$

- Proof: Using the linearity of expectation,

$$\begin{aligned}
\text{Var}(X) &= \text{E}((X - \mu)^2) \\
&= \text{E}(X^2 - 2\mu X + \mu^2) \\
&= \text{E}(X^2) - 2\mu\,\text{E}(X) + \text{E}(\mu^2) \\
&= \text{E}(X^2) - 2\mu^2 + \mu^2 \\
&= \text{E}(X^2) - \mu^2.
\end{aligned}$$

Example

- Suppose X is the result of the roll of one die.
- Recall: $\text{E}(X) = \frac{7}{2}$.
- Now

$$\text{E}(X^2) = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 25 \cdot \frac{1}{6} + 36 \cdot \frac{1}{6} = \frac{91}{6}.$$

- Hence

$$\text{Var}(X) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}.$$

Example

- Let $X \sim \text{Geom}(p)$ and let $q = 1 - p$.
- Recall: $\mathrm{E}(X) = \frac{q}{p}$ and $\mathrm{E}(X^2) = \frac{q(1+q)}{p^2}$.
- Thus

$$\mathrm{Var}(X) = \frac{q(1+q)}{p^2} - \frac{q^2}{p^2} = \frac{q}{p^2}.$$

Products of independent random variables

- Suppose X and Y are independent random variables with probability mass functions p_X and p_Y, respectively.
- Let $Z = XY$, and let p_Z be the probability mass function of Z.
- Let A, B, and C be the supports of X, Y, and Z, respectively.
- Then

$$\begin{aligned}
\mathrm{E}(Z) &= \sum_{z \in C} z p_Z(z) = \sum_{z \in C} z \sum_{\substack{x \in A, y \in B \\ xy = z}} \sum P(X = x, Y = y) \\
&= \sum_{z \in C} z \sum_{\substack{x \in A, y \in B \\ xy = z}} \sum p_X(x) p_Y(y) = \sum_{x \in A} \sum_{y \in B} xy p_X(x) p_Y(y) \\
&= \sum_{x \in A} x p_X(x) \sum_{y \in B} y p_Y(y) = \mathrm{E}(Y) \sum_{x \in A} x p_X(x) = \mathrm{E}(X)\,\mathrm{E}(Y).
\end{aligned}$$

Products of independent random variables

- That is: if X and Y are independent, then $\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y)$.

Variance of a sum

- Suppose X and Y are independent random variables with means μ_X and μ_Y.
- Recall: $\mathrm{E}(X + Y) = \mu_X + \mu_Y$.
- Note: $\mathrm{E}(X - \mu_X) = \mathrm{E}(X) - \mathrm{E}(\mu_X) = \mu_X - \mu_X = 0$.
- Hence

$$\begin{aligned}
\mathrm{Var}(X + Y) &= \mathrm{E}(((X + Y) - (\mu_X + \mu_Y))^2) = \mathrm{E}(((X - \mu_X) + (Y - \mu_Y))^2) \\
&= \mathrm{E}((X - \mu_X)^2 + 2(X - \mu_X)(Y - \mu_Y) + (Y - \mu_Y)^2) \\
&= \mathrm{E}((X - \mu_X)^2) + 2\,\mathrm{E}(X - \mu_X)\,\mathrm{E}(Y - \mu_Y) + \mathrm{E}((Y - \mu_Y)^2) \\
&= \mathrm{Var}(X) + \mathrm{Var}(Y).
\end{aligned}$$

Example

- Suppose a fair die is rolled twice.
- Let X be the result of the first roll, Y the result of the second roll, and $T = X + Y$.
- Then

$$\mathrm{Var}(T) = \mathrm{Var}(X) + \mathrm{Var}(Y) = \frac{35}{12} + \frac{35}{12} = \frac{35}{6}.$$

Example

- Suppose X is a Bernoulli random variable with probability of success p, $0 \leq p \leq 1$, and probability of failure $q = 1 - p$.
- That is, X has probability mass function

$$p_X(x) = \begin{cases} q, & \text{if } x = 0, \\ p, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Recall: we have seen that $\mathrm{E}(X) = p$.
- Now

$$\mathrm{E}(X^2) = 0 + p = p.$$

- So

$$\mathrm{Var}(X) = p - p^2 = p(1 - p) = pq.$$

Example

- Now let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables, each with probability of success p and probability of failure $q = 1 - p$.
- Let $Y = X_1 + X_2 + \cdots + X_n$.
- Then $Y \sim \mathrm{Bin}(n, p)$.
- And

$$\mathrm{Var}(Y) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n) = npq.$$

Multiples and translations

- Proposition: If X is a random variable and a and b are any constants, then

$$\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X).$$

- Proof: Letting $\mu = \mathrm{E}(X)$, and noting that $\mathrm{E}(aX + b) = a\mu + b$,

$$\begin{aligned} \mathrm{Var}(aX + b) &= \mathrm{E}(((aX + b) - (a\mu + b))^2) \\ &= \mathrm{E}((aX - a\mu)^2) \\ &= \mathrm{E}(a^2(X - \mu)^2) \\ &= a^2 \mathrm{E}((X - \mu)^2) \\ &= a^2 \mathrm{Var}(X). \end{aligned}$$

Example

- Suppose $X \sim \text{Geom}(p)$.
- Then $Y = X + 1 \sim \text{FS}(p)$.
- Hence $\text{Var}(Y) = \text{Var}(X) = \frac{q}{p^2}$.