Mathematics 340: Lecture 18

Continuous Random Variables

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Continuous random variables

- We say a random variable X is *continuous* if there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

- We call f the *probability density function* of X.
- Note:
  - For any real number a,

$$P(X = a) = P(a \leq X \leq a) = \int_a^a f(x)dx = 0.$$

  - Hence the range of a continuous random variable X must be uncountable.
  - And $P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b)$, for any real numbers $a \leq b$.

Probability density functions

- Note: If f is a probability density function of a continuous random variable X, then
  - $f(x) \geq 0$ for all $x \in \mathbb{R}$, and
  - $\int_{-\infty}^{\infty} f(x)dx = 1$.
- Conversely, any function $f : \mathbb{R} \to \mathbb{R}$ satisfying the above two properties is a probability density function for some random variable.
- Suppose F is the cumulative distribution function of a continuous random variable X with probability density function f. Then
  - $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$, and
  - $\frac{d}{dx}F(x) = f(x)$.
  - In particular, F is continuous.

Example

- Suppose X is a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{4}(1+x), & \text{if } 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

- Then, for example,

$$P\left(\frac{1}{2} < X < 1\right) = \int_{\frac{1}{2}}^{1} \frac{1}{4}(1+x)dx = \frac{7}{32}.$$

Example (cont'd)

- If F is the distribution function of X, then, for $0 \leq x \leq 2$,

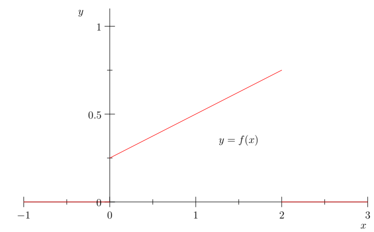
$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} \frac{1}{4}(1+t)dt = \frac{x}{4} + \frac{x^2}{8}.$$

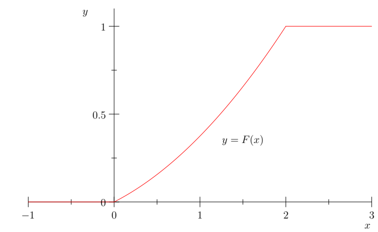
- Hence

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{x}{4} + \frac{x^2}{8}, & \text{if } 0 \leq x \leq 2, \\ 1, & \text{if } x > 2. \end{cases}$$

Example (cont'd)

- Graphs of $y = f(x)$ and $y = F(x)$:





Expected values

- Suppose X is a continuous random variable with probability mass function $f_X(x)$.
- If

$$\int_{-\infty}^{\infty} |x| f_X(x)dx < \infty,$$

then the *expected value* of X is

$$\text{E}(X) = \int_{-\infty}^{\infty} x f_X(x)dx.$$

- As with discrete random variables, if $g : \mathbb{R} \to \mathbb{R}$ and $Y = g(X)$,

$$\text{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)dx.$$

- And the *variance* of X is

$$\text{Var}(X) = \text{E}((X-\mu)^2) = \text{E}(X^2) - \mu^2,$$

where $\mu = \text{E}(X)$.

Example

- Suppose X is a random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{4}(1+x), & \text{if } 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

- Then

$$\text{E}(X) = \int_{-\infty}^{\infty} x f(x)dx = \frac{1}{4}\int_{0}^{2}(x+x^2)dx = \frac{1}{4}\left(2+\frac{8}{3}\right) = \frac{7}{6}.$$

- And

$$\text{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{4}\int_{0}^{2}(x^2+x^3)dx = \frac{1}{4}\left(\frac{8}{3}+4\right) = \frac{5}{3}.$$

- Thus $\text{Var}(X) = \frac{5}{3} - \frac{49}{36} = \frac{11}{36}$.

Uniform random variable

- Given any two real numbers $a < b$, we say a random variable X with probability density function
$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a < x < b, \\ 0, & \text{otherwise,} \end{cases}$$
has a *uniform* distribution on (a, b), denoted $X \sim \text{Unif}(a, b)$.
- Note: the uniform distribution models choosing a number at random from an interval.

Expected value and variance

- Suppose X has a uniform distribution on (a, b).
- Then
$$\text{E}(X) = \int_a^b \frac{x}{b-a}\,dx = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}.$$
- And
$$\text{E}(X^2) = \int_a^b \frac{x^2}{a-b}\,dx = \frac{b^3 - a^3}{3(a-b)} = \frac{b^2 + ab + a^2}{3}.$$
- So
$$\text{Var}(X) = \frac{b^2 + ab + a^2}{3} - \frac{(a+b)^2}{4} = \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}.$$

Example

- Suppose $X \sim \text{Unif}(-2, 2)$.
- Then the probability density function of X is
$$f_X(x) = \begin{cases} \frac{1}{4}, & \text{if } -2 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Example (cont'd)

- Let $Y = X^2$.
- If F_Y is the cumulative distribution function of Y, and $0 < y < 4$, then
$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{4}\,dx \\ &= \frac{\sqrt{y}}{2}. \end{aligned}$$

Example (cont'd)

- If f_Y is the probability density function for y, then, for $0 < y < 4$,

$$f_Y(y) = \frac{d}{dy}\left(\frac{\sqrt{y}}{2}\right) = \frac{1}{4\sqrt{y}}.$$

- Hence

$$f_Y(y) = \begin{cases} \dfrac{1}{4\sqrt{y}}, & \text{if } 0 < y < 4, \\ 0, \text{otherwise.} \end{cases}$$

Example (cont'd)

- Graphs of $z = f_X(x)$ and $z = f_Y(y)$: