Mathematics 340: Lecture 26

Covariance

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Covariance

- Suppose X and Y are random variables with means μ_X and μ_Y, respectively, and $E(|XY|) < \infty$. We call

$$\text{Cov}(X, Y) = \text{E}((X - \mu_X)(Y - \mu_Y))$$

  the *covariance* of X and Y.
- Note:
  - $\text{Cov}(X, X) = \text{Var}(X)$.
  - If X and Y are independent, then

$$\text{Cov}(X, Y) = \text{E}((X - \mu_X)(Y - \mu_Y)) = \text{E}(X - \mu_X)\,\text{E}(Y - \mu_Y) = 0.$$

- If $\text{Cov}(X, Y) = 0$, we say X and Y are *uncorrelated*.
- Note: independent random variables are uncorrelated, but uncorrelated random variable are not necessarily independent.

Calculation

- For any random variables X and Y with means μ_X and μ_Y, respectively,

$$\begin{aligned}\text{Cov}(X, Y) &= \text{E}((X - \mu_X)(Y - \mu_Y)) \\ &= \text{E}(XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y) \\ &= \text{E}(XY) - \mu_Y\,\text{E}(X) - \mu_X\,\text{E}(Y) + \mu_X\mu_Y \\ &= \text{E}(XY) - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y \\ &= \text{E}(XY) - \mu_X\mu_Y.\end{aligned}$$

Example

- Suppose X and Y have the joint density $f(x, y) = \begin{cases} 10x^2y, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$
- Then

$$\begin{aligned}\text{E}(X) &= \int_0^1 \int_0^x 10x^3y\,dydx = \frac{5}{6}, \\ \text{E}(Y) &= \int_0^1 \int_0^x 10x^2y^2\,dydx = \frac{5}{9}, \\ \text{E}(XY) &= \int_0^1 \int_0^x 10x^3y^2\,dydx = \frac{10}{21}.\end{aligned}$$

- Hence

$$\text{Cov}(X, Y) = \frac{10}{21} - \left(\frac{5}{6}\right)\left(\frac{5}{9}\right) = \frac{5}{378} = 0.01323.$$

- Note: since $\text{Cov}(X, Y) \neq 0$, this is another way to see that X and Y are not independent.

Variance and covariance

- Suppose X and Y are random variables with mean μ_X and μ_Y, respectively.
- Then, since $\mathrm{E}(X+Y) = \mu_X + \mu_Y$,

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{E}(((X+Y)-(\mu_X+\mu_Y))^2)\\ &= \mathrm{E}(((X-\mu_X)+(Y-\mu_Y))^2)\\ &= \mathrm{E}((X-\mu_X)^) + 2\,\mathrm{E}((X-\mu_X)(Y-\mu_Y)) + \mathrm{E}((Y-\mu_Y)^2)\\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y).\end{aligned}$$

- More generally, for random variables $X_1, X_2, \ldots, X_n$,

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \sum_{k=1}^{n} \mathrm{Var}(X_k) + 2\sum_{1\le i<j\le n} \mathrm{Cov}(X_i, X_j).$$

Example

- Suppose X and Y have the joint density $f(x,y) = \begin{cases} 10x^2y, & \text{if } 0 < y < x < 1,\\ 0, & \text{otherwise.}\end{cases}$
- Then

$$E(X^2) = \int_0^1 \int_0^x 10x^4 y dy dx = \frac{5}{7},$$

and

$$E(Y^2) = \int_0^1 \int_0^x 10x^2 y^3 dy dx = \frac{5}{14}.$$

- Then, from the previous example,

$$\mathrm{Var}(X) = \frac{5}{7} - \frac{25}{36} = \frac{5}{252} \text{ and } \mathrm{Var}(Y) = \frac{5}{14} - \frac{25}{81} = \frac{55}{1134}.$$

- Hence

$$\mathrm{Var}(X+Y) = \frac{5}{252} + \frac{55}{1134} + 2\cdot\frac{5}{378} = \frac{215}{2268} = 0.09480.$$

Cauchy-Schwarz inequality

- Suppose X and Y are random variables.
- For any real number t,

$$0 \le \mathrm{E}((X-tY)^2) = \mathrm{E}(X^2) - 2tE(XY) + t^2E(Y^2).$$

- Note: the right-hand side of the above is a quadratic polynomial, in the variable t, which can have at most one root
- Hence, using the quadratic formula, it must be the case that

$$4(\mathrm{E}(XY))^2 - 4E(X^2)E(Y^2) \le 0.$$

- That is $(\mathrm{E}(XY))^2 \le \mathrm{E}(X^2)\,\mathrm{E}(Y^2)$, or $|\,\mathrm{E}(XY)| \le \sqrt{\mathrm{E}(X^2)\,\mathrm{E}(Y^2)}$.

Correlation coefficient

- Suppose X and Y are random variable with means μ_X and μ_Y and variances σ_X^2 and σ_Y^2, respectively.
- Applying the previous result to the random variables $X - \mu_X$ and $Y - \mu_Y$, we have

$$|\,\mathrm{Cov}(X,Y)| \le \sigma_X\sigma_Y.$$

- It follows that

$$-1 \le \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} \le 1.$$

- We all

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}$$

the *correlation coefficient* of X and Y.
- Notation: $\mathrm{Corr}(X,Y)$ is frequently denoted ρ.
- Note: $\mathrm{Corr}(X,Y) = 1$ or $\mathrm{Corr}(X,Y) = -1$ if and only if $Y = aX + b$ for some constants a and b.

Example

- Suppose X and Y have the joint density $f(x,y) = \begin{cases} 10x^2y, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$
- Then

$$\mathrm{Corr}(X,Y) = \frac{\frac{5}{378}}{\sqrt{\frac{5}{252}}\sqrt{\frac{55}{1134}}} = 0.4264.$$