Mathematics 340: Lecture 27

The Gamma Distribution

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Gamma function

- For any $\alpha > 0$, we define
$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx.$$
- We call Γ the *gamma function*.
- For example,
$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1.$$

Example

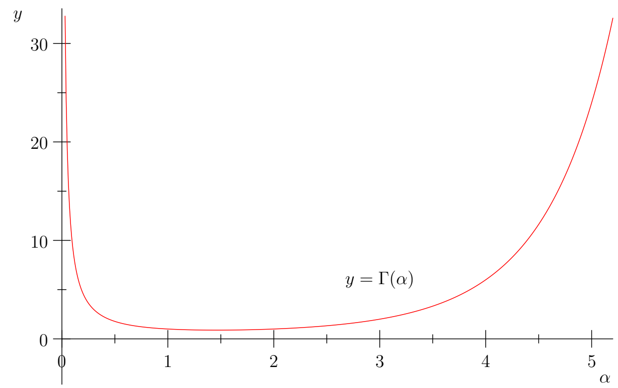
- Another example: using the substitution $u = \sqrt{x}$,
$$\begin{aligned}\Gamma\left(\tfrac{1}{2}\right) &= \int_0^\infty \frac{1}{\sqrt{x}}e^{-x}dx \\ &= 2\int_0^\infty e^{-u^2}du \\ &= \int_{-\infty}^\infty e^{-u^2}du \\ &= \sqrt{\pi}.\end{aligned}$$

Factorials

- If $\alpha > 1$, then, using integration by parts,
$$\begin{aligned}\Gamma(\alpha) &= \int_0^\infty x^{\alpha-1}e^{-x}dx \\ &= \lim_{b\to\infty}\left(-x^{\alpha-1}e^{-x}\Big|_0^b + (\alpha-1)\int_0^b x^{\alpha-2}e^{-x}dx\right) \\ &= -\lim_{b\to\infty} b^{\alpha-1}e^{-b} + (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}dx \\ &= (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}dx \\ &= (\alpha-1)\Gamma(\alpha-1).\end{aligned}$$
- Hence, if $n > 1$ is an integer,
$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\cdots(2)(1)\Gamma(1) = (n-1)!.$$

Gamma function

- Graph of $y = \Gamma(t)$:



Gamma random variables

- Given $\lambda > 0$ and $\alpha > 0$, we say a random variable X with probability density function

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

has a *gamma distribution*, and write $X \sim \text{Gamma}(\alpha, \lambda)$.

- Note: using the substitution $u = \lambda x$,

$$\int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\lambda} \left(\frac{u}{\lambda}\right)^{\alpha-1} e^{-u} du$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} du = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1.$$

- Note: if $\alpha = 1$, then X has an exponential distribution.

Waiting times

- Let N_t be a Poisson process with rate λ.
- That is, N_t is the number of events which have occurred up to time t, where the number of events has a Poisson distribution with rate λ per unit of time.
- Recall: if T_n is the time at which the nth event occurs, and F_{T_n} is the cumulative distribution funciton of T_N, then, for $t > 0$,

$$F_{T_n}(t) = 1 - P(T_n > t)$$
$$= 1 - P(N_t \leq n - 1)$$
$$= 1 - \sum_{k=0}^{n-1} \frac{\lambda^k t^k e^{-\lambda t}}{k!}.$$

Waiting times (cont'd)

- If f_{T_n} is the probability density function of T_N, then, for $t \geq 0$,

$$f_{T_n}(t) = -\frac{d}{dt} \sum_{k=0}^{n-1} \frac{\lambda^k t^k e^{-\lambda t}}{k!} = \lambda e^{-\lambda t} - \sum_{k=1}^{n-1} \left(-\frac{\lambda^{k+1} t^k e^{-\lambda t}}{k!} + \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} \right)$$
$$= \lambda e^{-\lambda t} + \left(\frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} - \lambda e^{-\lambda t} \right) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}.$$

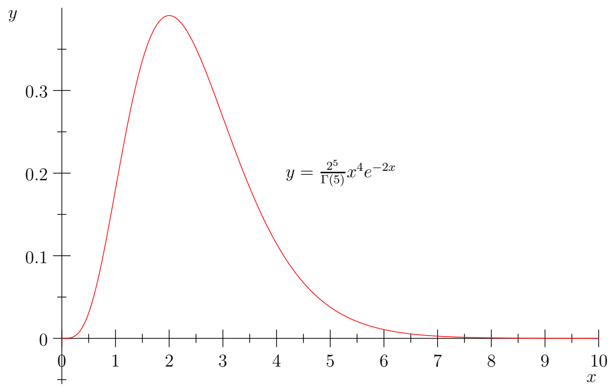
- That is, T_n has the density

$$f_{T_n}(t) = \begin{cases} \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t}, & \text{if } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Hence T_n has a gamma distribution with parameters n and λ.

Graph of gamma density

- Graph of a gamma density with $\lambda = 2$ and $\alpha = 5$:



Expectation

- Suppose X is a gamma random variable with parameters α and λ.
- Then

$$E(X) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\lambda x} dx = \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} \int_0^\infty \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-\lambda x} dx = \frac{\alpha}{\lambda}.$$

- And

$$E(X^2) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha+1} e^{-\lambda x} dx = \frac{\Gamma(\alpha+2)}{\lambda^2\Gamma(\alpha)} \int_0^\infty \frac{\lambda^{\alpha+2}}{\Gamma(\alpha+2)} x^{\alpha+1} e^{-\lambda x} dx = \frac{\alpha(\alpha+1)}{\lambda^2}.$$

- Hence

$$\mathrm{Var}(X) = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}.$$

Moment generating function

- The moment generating function of X is

$$\begin{aligned} M_X(t) &= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{tx} x^{\alpha-1} e^{-\lambda x} dx \\ &= \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \int_0^\infty \frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx \\ &= \left(\frac{\lambda}{\lambda-t}\right)^\alpha, \end{aligned}$$

provided $t < \lambda$.

Sum of exponentials

- Recall: if $X \sim \mathrm{Expo}(\lambda)$, then the moment generating function of X is

$$M_X(t) = \frac{\lambda}{\lambda - t}, \text{ for } t < \lambda.$$

- Let $X_1, X_2, \ldots, X_n$ be independent random variables, each having an exponential distribution with parameter λ.
- Let $S_n = X_1 + X_2 + \cdots + X_n$.
- Then the moment generating function of S is

$$M_{S_n}(t) = \left(\frac{\lambda}{\lambda-t}\right)^n, \text{ for } t < \lambda.$$

- It follows that $S_n \sim \mathrm{Gamma}(n, \lambda)$.