Mathematics 340: Lecture 28

Some Inequalities

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Markov's inequality

- Suppose X is a nonnegative random variable and $a > 0$.
- Then
$$P(X \geq a) \leq \frac{\mathrm{E}(X)}{a}.$$
- Proof:
  - Suppose X is a continuous random variable with probability density function f_X.
  - Then
$$\mathrm{E}(X) = \int_0^\infty x f_X(x)dx \geq \int_a^\infty x f_X(x)dx \geq \int_a^\infty a f_X(x)dx = aP(X \geq a).$$
- Note: it follows that, for any random variable X and constant $a > 0$,
$$P(|X| \geq a) \leq \frac{\mathrm{E}(|X|)}{a}.$$

Example

- Suppose X is a nonnegative random variable with mean μ_X.
- Then, for any $k > 0$,
$$P(X \geq k\mu_X) \leq \frac{\mu_X}{k\mu_X} = \frac{1}{k}.$$
- For example, $P(X \geq 2\mu_X) \leq \frac{1}{2}$ and $P(X \geq 4\mu_X) \leq \frac{1}{4}$.
- Note: Suppose $X \sim \text{Expo}(\lambda)$. Then
  - $P\left(X \geq \frac{2}{\lambda}\right) = e^{-2} = 0.1353$.
  - $P\left(X \geq \frac{4}{\lambda}\right) = e^{-4} = 0.01832$.

Chernoff's inequality

- Let $t > 0$.
- Note: If X is any random variable, then $Y = e^{tX}$ is a nonnegative random variable.
- Hence for any $a > 0$, by Markov's inequality,
$$P(Y \geq e^{ta}) \leq \frac{\mathrm{E}(Y)}{e^{ta}}.$$
- Note: $Y \geq e^{ta}$ if and only if $tX \geq ta$, that is, if and only if $X \geq a$.
- Hence we have
$$P(X \geq a) \leq \frac{E(e^{tX})}{e^{ta}},$$
  which is *Chernoff's inequality*.
- Note: $\mathrm{E}(e^{tX}) = M_X(t)$, the moment generating function of X.

Example

- Suppose $X \sim \text{Expo}(\lambda)$.
- Then, for any $0 < t < \lambda$,

$$P\left(X \geq \frac{4}{\lambda}\right) \leq \frac{M_X(t)}{e^{\frac{4t}{\lambda}}} = \frac{\lambda}{(\lambda - t)e^{\frac{4t}{\lambda}}}.$$

- Now $g(t) = \frac{\lambda}{(\lambda - t)e^{\frac{4t}{\lambda}}}$ is minimized, on the interval $(0, \lambda)$, when $t = \frac{3\lambda}{4}$.
- And $g\left(\frac{3\lambda}{4}\right) = 4e^{-3} = 0.1991$.
- Hence

$$\left(X \geq \frac{4}{\lambda}\right) \leq 0.1991.$$

Chebyshev's inequality

- Suppose X is a random variable with mean μ_X and variance σ_X^2. For any $a > 0$,

$$P(|X - \mu_X| \geq a) \leq \frac{\sigma_X^2}{a^2}.$$

- Proof:
  - From Markov's inequality,

$$P((X - \mu_X)^2 \geq a^2) \leq \frac{\text{E}((X - \mu_X)^2)}{a^2} = \frac{\sigma_X^2}{a^2}.$$

  - Note: $(X - \mu_X)^2 \geq a^2$ if and only if $|X - \mu_X| \geq a$.
  - Hence we have

$$P(|X - \mu_X| \geq a) \leq \frac{\sigma_X^2}{a^2}.$$

Example

- Suppose X is a random variable with mean 100 and variance 50.
- Then, for example, using Chebyshev's inequality,

$$P(|X - 100| \geq 50) \leq \frac{50}{(50)^2} = \frac{1}{50}.$$

- Similarly,

$$P(50 < X < 150) = P(|X - 100| < 50) = 1 - P(|X - 100| \geq 50) \geq 1 - \frac{1}{50} = \frac{49}{50}.$$

Example

- Suppose Z is standard normal.
- Using Markov's inequality,

$$P(|Z| \geq 3) \leq \frac{E(|Z|)}{3} = \frac{\sqrt{\frac{2}{\pi}}}{3} = \frac{1}{3}\sqrt{\frac{2}{\pi}} = 0.2660.$$

- Using Chebyshev's inequality, $P(|Z| \geq 3) \leq \frac{1}{9} = 0.1111$.
- Using Chernoff's inequality, and letting M_Z be the moment generating function of Z,

$$P(|Z| \geq 3) = 2P(Z \geq 3) \leq 2 \cdot \frac{M_Z(t)}{e^{3t}} = 2e^{-3t}e^{\frac{t^2}{2}}.$$

- Now $f(t) = -3t + \frac{1}{2}t^2$ is minimized when $t = 3$.
- Hence $P(|Z| \geq 3) \leq 2e^{-\frac{9}{2}} = 0.02222$.
- Note: exactly, $P(|Z| \geq 3) = 2(1 - \Phi(3)) = 2(1 - 0.9987) = 0.0026$.

Example

- Suppose X is a random variable with mean μ_X and variance σ_X^2.
- Using Chebyshev's inequality, for any $k > 0$,

$$P(|X - \mu_X| \geq k\sigma_X) \leq \frac{\sigma_X^2}{k^2\sigma^2} = \frac{1}{k^2}.$$