Mathematics 340: Lecture 7

Bayes' Theorem

Dan Sloughter

Furman University

September 11, 2015

Partitions

- We say events $A_1, A_2, \ldots, A_n$ *partition* a sample space Ω if $A_1, A_2, \ldots, A_n$ are mutually exclusive and
$$\Omega = \bigcup_{i=1}^{n} A_i.$$
- We also say that $A_1, A_2, \ldots, A_n$ are mutually exclusive and *exhaustive*.
- Note: If $A_1, A_2, \ldots, A_n$ partition Ω and B is any event in Ω, then
$$B = B \cap \Omega = B \cap (A_1 \cup A_2 \cup \cdots \cup A_n) = \bigcup_{i=1}^{n} B \cap A_i.$$

Law of total probability

- Suppose $A_1, A_2, \ldots, A_n$ is a partition of a sample space Ω and $B \subset \Omega$ is an event. Then
$$P(B) = \sum_{i=1}^{n} P(B \mid A_i)P(A_i).$$
- Proof:
$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B \mid A_i)P(A_i).$$
- This is the *law of total probability*.

Example

- Experiment: Roll a fair die, and then flip a fair coin the number of times shown on the die.
- Let B be the event the coin tosses result in 3 heads.
- Let A_i be the event that the roll of the die is i, $i = 1, 2, 3, 4, 5, 6$.
- Then
$$\begin{aligned} P(B) &= \sum_{i=1}^{6} P(B \mid A_i)P(A_i) \\ &= 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + \frac{1}{8} \cdot \frac{1}{6} + \frac{4}{16} \cdot \frac{1}{6} + \frac{10}{32} \cdot \frac{1}{6} + \frac{20}{64} \cdot \frac{1}{6} \\ &= \frac{1}{6}. \end{aligned}$$

Example

- Abel and Baker take turns rolling two dice.
- Abel goes first, and wins if he rolls a 6 before Baker rolls a 7; Baker wins if he rolls a 7 before Abel rolls a 6.
- What is the probability that Abel wins?
- Let A_1 be the event the first roll is a 6, A_2 the event the first roll is not a six and the second roll is a 7, and A_3 the event the first roll is not a 6 and the second roll is not a 7.
- Note: A_1, A_2, and A_3 partition the sample space.
- Let B be the event Abel wins.

Example (cont'd)

- Then

$$\begin{aligned} P(B) &= P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + P(B \mid A_3)P(A_3) \\ &= \left(1 \cdot \frac{5}{36}\right) + \left(0 \cdot \frac{31 \cdot 6}{36 \cdot 36}\right) + \left(P(B) \cdot \frac{31 \cdot 30}{36 \cdot 36}\right) \\ &= \frac{5}{36} + P(B) \cdot \frac{155}{216}. \end{aligned}$$

- Hence

$$P(B) = \frac{\frac{5}{36}}{\frac{61}{216}} = \frac{30}{61}.$$

Bayes' theorem

- Suppose $A_1, A_2, \ldots, A_n$ is a partition of a sample space Ω and $B \subseteq \Omega$ is an event. Then, for any $1 \leq i \leq n$,

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j)P(A_j)}.$$

- Proof:

$$P(A_i \mid B) = \frac{P(B \cap A_i)}{P(B)} = \frac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j)P(A_j)}.$$

- Note: For any event A, we have

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)}.$$

Example

- Suppose that it is known that 1 out of 10,000 people in the general population has a certain disease.
- Suppose there is a testing procedure which is known to correctly identify a person with the disease 99% of the time, but 5% of the time will give a positive result for a person who does not have the disease.
- What is the probability that a person who tests positive has the disease?
- Let A be the event a person has the disease and let B be the event that the test gives a positive result.

Example (cont'd)

- Then

$$\begin{aligned} P(A \mid B) &= \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} \\ &= \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.05)(0.9999)} \\ &= 0.001976. \end{aligned}$$

- Notes:
  - This low probability explains why widespread testing for a rare disease is usually discouraged.
  - Although it is still very unlikely that the person has the disease, it is far more likely that she or he has the disease than it was before knowing the test result.

Example

- Suppose we have 11 urns, numbered 0 through 10.
- Suppose the ith urn has i red chips and $10 - i$ white chips, $i = 0, 1, 2, \ldots, 10$.
- Experiment: An urn is selected at random, and then two chips are selected at random from that urn.
- Question: What is the probability that both chips selected are red?
- Let A_i, $i = 0, 1, 2, \ldots, 10$, be the event the ith urn is selected.
- Let B be the event that both chips selected are red.
- Then

$$P(B) = \sum_{i=0}^{10} P(B \mid A_i)P(A_i) = \frac{1}{11}\sum_{i=2}^{10} \frac{i(i-1)}{10 \cdot 9} = \frac{1}{11} \cdot \frac{11}{3} = \frac{1}{3}.$$

Example (cont'd)

- Question: If both chips selected are red, what is the probability that we are sampling from the 10th urn?
- Now we want

$$P(A_{10} \mid B) = \frac{P(B \mid A_{10})P(A_{10})}{\sum_{i=0}^{10} P(B \mid A_i)P(A_i)} = \frac{1 \cdot \frac{1}{11}}{\frac{1}{11}\sum_{i=2}^{10} \frac{i(i-1)}{10 \cdot 9}} = \frac{\frac{1}{11}}{\frac{1}{3}} = \frac{3}{11}.$$

Example (cont'd)

- Question: If the two chips selected are red, what is the probability that the next chip selected from the same urn is red?
- Let C be the event that a third chip selected is red.
- Then

$$\begin{aligned} P(C \mid B) = \frac{P(C \cap B)}{P(B)} &= \frac{\sum_{i=0}^{10} P(C \cap B \mid A_i)P(A_i)}{P(B)} \\ &= \frac{\frac{1}{11}\sum_{i=3}^{10} \frac{i(i-1)(i-2)}{10 \cdot 9 \cdot 8}}{P(B)} \\ &= \frac{\frac{1}{11} \cdot \frac{11}{4}}{\frac{1}{3}} = \frac{3}{4}. \end{aligned}$$