Mathematics 350: Lecture 17

The Modulus of an Integral

Dan Sloughter

Furman University

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Proposition

- If $|f(z)| \leq M$ for all $z \in C$, where C is a contour with parametrization $z(t)$, $a \leq t \leq b$, and L is the length of C, then
$$\left| \int_C f(z)dz \right| \leq ML.$$

Proof

- Since
$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt,$$
we have
$$\begin{aligned} \left| \int_C f(z)dz \right| &\leq \int_a^b |f(z(t))||z'(t)|dt \\ &\leq \int_a^b M|z'(t)|dt \\ &= M\int_a^b |z'(t)|dt \\ &= ML. \end{aligned}$$
- Note: Such an M always exists since we assume that $f(z(t))$ is a piecewise continuous function on a closed interval $[a, b]$ (this is the extreme value theorem from calculus).

Example

- Consider
$$\int_C \frac{z^2+1}{z^6+1}dz$$
where C is the arc of the circle $|z| = 3$ from 3 to -3.
- Now for z on C,
$$|z^2+1| \leq |z^2| + |1| = |z|^2 + 1 = 9 + 1 = 10$$
and
$$|z^6+1| \geq ||z|^6 - |1|| = 728.$$
- Hence
$$\left| \frac{z^2+1}{z^6+1} \right| = \frac{|z^2+1|}{|z^6+1|} \leq \frac{10}{728} = \frac{5}{364}.$$

Example

- Since C has length 3π, it follows that
$$\left|\int_C \frac{z^2+1}{z^6+1}dz\right| \leq \frac{15\pi}{364}.$$
- Now consider
$$\int_C \frac{z^2+1}{z^6+1}dz$$
where C is the arc of the circle $|z| = R$ from R to $-R$, $R > 1$.
- We have, for z on C,
$$|z^2+1| \leq R^2+1$$
and
$$|z^6+1| \geq R^6-1.$$

Example

- So
$$\left|\frac{z^2+1}{z^6+1}\right| \leq \frac{R^2+1}{R^6-1}.$$
- Thus
$$\left|\int_C \frac{z^2+1}{z^6+1}dz\right| \leq \frac{(R^2+1)R\pi}{R^6-1}.$$
- Now
$$\lim_{R\to\infty} \frac{(R^2+1)R\pi}{R^6-1} = \lim_{R\to\infty} \frac{\frac{\pi}{R^3}+\frac{\pi}{R^5}}{1-\frac{1}{R^6}} = 0.$$
- So
$$\lim_{R\to\infty} \left|\int_C \frac{z^2+1}{z^6+1}dz\right| = 0.$$
- Hence $\lim_{R\to\infty} \int_C \frac{z^2+1}{z^6+1}dz = 0.$