Mathematics 350: Lecture 20

Simply and Multiply Connected Domains

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Simply connected domains

- We say a domain D is *simply connected* if, whenever $C \subset D$ is a simple closed contour, every point in the interior of C lies in D.
- We say a domain which is not simply connected is *multiply connected*
- Examples
  - The domain
$$U = \{z \in \mathbb{C} : |z| < 1\}$$
    is simply connected.
  - The domain
$$A = \{z \in \mathbb{C} : 1 < |z| < 2\}$$
    is not simply connected.

Theorem

- If D is a simply connected domain and f is analytic in D, then
$$\int_C f(z)dz = 0$$
  for every closed contour C in D.
- Proof
  - If C is a simple closed contour, then the conclusion follows from the Cauchy-Goursat theorem.
  - If C is not simple, but intersects itself only a finite number of times, then the conclusion follows by writing C as a sum of simple closed contours.
  - We will omit the more difficult situation in which C intersects itself an infinite number of times.

Corollary

- If D is a simply connected domain and f is analytic in D, then f has an antiderivative at all points of D.
- Note: In particular, entire functions have antiderivatives on all of $\mathbb{C}$.

Theorem

- Suppose C is a positively oriented, simple closed contour and that C_1, C_2, $\ldots C_n$ are negatively oriented, simple closed contours, all of which are in the interior of C, are disjoint, and have disjoint interiors.
- Let R be the region consisting of C, C_1, C_2, $\ldots$, C_n, and all points which are in the interior of C and the exterior of each C_k.
- If f is analytic in R, then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0.$$

Proof

- Let L_1 be a polygonal path connecting C to C_1, L_k a polygonal path connecting C_{k-1} to C_k, $k = 2, 3, \ldots, n$, and L_{n+1} a polygonal path connecting C_n to C.
- Let B_1 be the part of C from where L_{n+1} joins C to where L_1 joins C, B_2 the remaining part of C, α_k the part of C_k between where L_k and L_{k+1} join C_k, and β_k the remaining part of C_k.
- Let

$$\Gamma_1 = B_1 + L_1 + \alpha_1 + L_2 + \alpha_2 + \cdots + \alpha_n + L_{n+1}$$

and

$$\Gamma_2 = B_2 - L_{n+1} + \beta_n - L_n + \beta_{n-1} - \cdots + \beta_1 - L_1.$$

Proof (cont'd)

- Then, by the Cauchy-Goursat theorem,

$$\int_{\Gamma_1} f(z)dz = 0 = \int_{\Gamma_2} f(z)dz.$$

- Hence

$$0 = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = \int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz.$$

Corollary

- Suppose C_1 and C_2 are positively oriented, simply closed contours with C_2 lying in the interior of C_1.
- Let R be the region consisting of C_1, C_2, and the part of the interior of C_1 which is in the exterior of C_2.
- If f is analytic in R, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

- Proof: From the previous theorem, we have

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0.$$

Example

- By a homework exercise, if C_0 is any positively oriented circle with center at the origin, then
$$\int_{C_0} \frac{1}{z} dz = 2\pi i.$$
- It now follows that if C is any positively oriented, simple closed contour with the origin in its interior, then
$$\int_{C} \frac{1}{z} dz = 2\pi i.$$