Mathematics 350: Lecture 29

Cauchy's Residue Theorem

Dan Sloughter

Furman University

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Cauchy's residue theorem

- Suppose C is a positively oriented, simple closed contour.
- If f is analytic on and inside C except for the finite number of singular points z_1, z_2, ..., z_n, then
$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Example

- Suppose C is the circle $|z| = 2$ with positive orientation and we wish to evaluate
$$\int_C \frac{1}{(z+1)(z-1)^3} dz.$$
- In a previous example we saw that
$$\frac{1}{(z+1)(z-1)^3} = -\frac{1}{8(z+1)} + \frac{1}{8(z-1)} - \frac{1}{4(z-1)^2} + \frac{1}{2(z-1)^3}.$$
- It follows that
$$\operatorname*{Res}_{z=-1} \frac{1}{(z+1)(z-1)^3} = -\frac{1}{8}$$
and
$$\operatorname*{Res}_{z=1} \frac{1}{(z+1)(z-1)^3} = \frac{1}{8}.$$

Example (cont'd)

- Hence
$$\int_C \frac{1}{(z+1)(z-z)^3} dz = 2\pi i \left(-\frac{1}{8} + \frac{1}{8}\right) = 0.$$

Example

- Suppose C is the circle $|z| = 2$ with positive orientation and we wish to evaluate
$$\int_C \frac{3z+5}{z^2+1}dz.$$
- Using partial fractions,
$$\frac{3z+5}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i} = \frac{A(z-i)+B(z+i)}{(z+i)(z-i)}$$
for some constants A and B.
- Hence
$$3z+5 = A(z-i)+B(z+i).$$
- When $z=-i$, we have $5-3i=-2iA$ and when $z=i$ we have $5+3i=2iB$.
- Hence
$$A = \frac{3}{2}+\frac{5}{2}i \text{ and } B = \frac{3}{2}-\frac{5}{2}i.$$

Example (cont'd)

- Thus
$$\frac{3z+5}{z^2+1} = \frac{\frac{3}{2}+\frac{5}{2}i}{z+i} + \frac{\frac{3}{2}-\frac{5}{2}i}{z-i}.$$
- And so
$$\operatorname*{Res}_{z=-i}\frac{3z+5}{z^2+1} = \frac{3}{2}+\frac{5}{2}i$$
and
$$\operatorname*{Res}_{z=i}\frac{3z+5}{z^2+1} = \frac{3}{2}-\frac{5}{2}i.$$
- Hence
$$\int_C \frac{3z+5}{z^2+1}dz = 2\pi i\left(\left(\frac{3}{2}+\frac{5}{2}i\right)+\left(\frac{3}{2}-\frac{5}{2}i\right)\right) = 6\pi i.$$

Theorem

- Suppose f is analytic everywhere in the plane except at a finite number of singular points.
- If C is a positively oriented, simple closed contour containing all the singular points of f, then
$$\int_C f(z)dz = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right).$$
- We call $\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right)$ the *residue at infinity*.

Proof

- Let $R_1 > 0$ be such that the circle $|z| = R_1$ contains C and let C_0 be the circle $|z| = R_0$ where $R_0 > R_1$.
- We first note that
$$\int_{C_0} f(z)dz = \int_C f(z)dz.$$
- Moreover, we know that $f(z)$ has a Laurent series representation on the domain $|z| > R_1$.
- That is, for $|z| > R_1$,
$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n,$$
where
$$c_n = \frac{1}{2\pi i}\int_{C_0}\frac{f(z)}{z^{n+1}}dz.$$

Proof (cont'd)

- In particular,

$$2\pi i c_{-1} = \int_{C_0} f(z)dz.$$

- Now

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}}$$

for

$$\frac{1}{|z|} > R_1, \text{ that is } 0 < |z| < \frac{1}{R_1}.$$

- Hence

$$c_{-1} = \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right).$$

Example

- If

$$f(z) = \frac{1}{(z+1)(z-1)^3},$$

then

$$f\left(\frac{1}{z}\right) = \frac{1}{\left(\frac{1}{z}+1\right)\left(\frac{1}{z}-1\right)^3} = -\frac{z^4}{(z+1)(z-1)^3}.$$

- And so

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = -\frac{z^2}{(z+1)(z-1)^3}.$$

- Since this function is analytic at $z = 0$, we have

$$\operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right) = 0.$$

Example (cont'd)

- Hence if C is the circle $|z| = 2$, then

$$\int_C \frac{1}{(z+1)(z-1)^3} dz = 0.$$

Example

- If

$$f(z) = \frac{3z+5}{z^2+1},$$

then

$$f\left(\frac{1}{z}\right) = \frac{\frac{3}{z}+5}{\frac{1}{z^2}+1} = \frac{3z+5z^2}{1+z^2}.$$

- And so

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{3+5z}{z(1+z^2)}.$$

- Now

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$

for $|z| < 1$.

Example (cont'd)

- And so
$$\frac{3+5z}{1+z^2} = 3 + 5z - 3z^2 - 5z^3 + 3z^4 + 5z^5 - \cdots$$
for $|z| < 1$.
- Thus
$$\frac{3+5z}{z(1+z^2)} = \frac{3}{z} + 5 - 3z - 5z^2 + 3z^3 + 5z^4 - \cdots .$$
- Hence if C is the circle $|z| = 2$, with positive orientation,
$$\int_C \frac{3z+5}{z^2+1} dz = 6\pi i.$$