Mathematics 450: Lecture 10

Compact Spaces

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Definitions

- If S is a subset of a metric space E and, for some set A, $\{U_\alpha : \alpha \in A\}$ is a collection of open subsets of E, we say $\{U_\alpha : \alpha \in A\}$ is an *open cover* of S if

$$S \subset \bigcup_{\alpha \in A} U_\alpha.$$

- If $\{U_\alpha : \alpha \in A\}$ is an open cover of a set S in a metric space E, we call a subset of $\{U_\alpha : \alpha \in A\}$ which is also an open cover of S a *subcover* of S.
- We say a subset S of a metric space E is *compact* if every open cover S has a finite subcover of S.
- Examples:
  - Any finite subset of a metric space is compact.
  - If $S = (0,1)$ in $\mathbb{R}$, then $\left\{\left(\frac{1}{n},1\right) : n = 2,3,4,\ldots\right\}$ is an open cover of S which does not have a finite subcover of S.

Proposition

- Proposition: If E is compact and S is a closed subset of E, then S is compact.
- Proof:
  - Suppose $\{U_\alpha : \alpha \in A\}$ is an open cover of S.
  - Since S^c is an open set, $\{U_\alpha : \alpha \in A\} \cup \{S^c\}$ is an open cover of E.
  - Since E is compact, there exists a finite subset B of A for which $\{U_\alpha : \alpha \in B\} \cup \{S^c\}$ is a finite subcover of E.
  - Hence $\{U_\alpha : \alpha \in B\}$ is a finite subcover of S.
  - Thus S is compact.

Proposition

- Proposition: A compact set is bounded.
- Proof:
  - Suppose S is a compact subset of a metric space E.
  - For $p \in S$, let U_p be the open ball of radius 1.
  - The $\{U_p : p \in S\}$ is an open cover of S.
  - Since S is compact, there exists a finite set of points $p_1, p_2, \ldots, p_n$ such that $\{U_{p_i} : i = 1,2,\ldots,n\}$ is a finite subcover of S.
  - Let r be the maximum of $d(p_1, p_i)$, $i = 2,3,\ldots,n$.
  - Then S is a subset of the closed ball with center p_1 and radius $r+1$.
  - Hence S is bounded.

Nested set property

- Theorem:
  - Suppose $S_1 \supset S_2 \supset S_3 \supset \cdots$ is a nested sequence of nonempty closed subsets of a compact metric space E.
  - Then
$$\bigcap_{i=1}^{\infty} S_i \neq \emptyset.$$

Proof

- Suppose $\bigcap_{i=1}^{\infty} S_i = \emptyset$.
- Then $\{S_i^c : i = 1, 2, 3, \ldots\}$ is an open open cover of E.
- Since E is compact, there exist a finite list of indices n_1, n_2, ..., n_m, in increasing order, such that
$$E \subset \bigcup_{i=1}^{m} S_{n_i}^c.$$
- Now $S_{n_1}^c \subset S_{n_2}^c \subset \cdots \subset S_{n_m}^c$.
- Hence
$$E \subset \bigcup_{i=1}^{m} S_{n_i}^c = S_{n_m}^c.$$
- But then $E = S_{n_m}^c$, so $S_{n_m} = \emptyset$, contradicting our assumptions.

Example

- In $\mathbb{R}$, $S_n = [n, \infty)$, $n = 1, 2, 3, \ldots$, is a nested sequence of closed sets.
- Then $S_1 \supset S_2 \supset S_3 \supset \cdots$, but
$$\bigcap_{n=1}^{\infty} S_n = \emptyset.$$

Definition

- Definition:
  - Suppose S is a subset of a metric space E.
  - We say a point $p \in S$ is a *cluster point* of S if every open ball with center p contains an infinite number of points of S.
- Examples:
  - In $\mathbb{R}$, let $p_n = (-1)^n + \frac{1}{n}$, $n = 1, 2, 3, \ldots$, and let $S = \{p_1, p_2, p_3, \ldots\}$. Then -1 and 1 are cluster points of S.
  - Every $x \in \mathbb{R}$ is a cluster point of $\mathbb{Q}$.

Theorem

- Theorem: An infinite subset of a compact metric space has at least one cluster point.
- Proof:
  - Suppose S is an infinite subset of a compact metric space E.
  - Suppose S does not have a cluster point.
  - For every $p \in E$, let U_p be an open ball with center p which contains at most a finite number of points of S.
  - Then $\{U_p : p \in E\}$ is an open cover of E, and so has a finite subcover.
  - Let $p_1, p_2, \ldots, p_n$ be points of E for which $\{U_{p_i} : i = 1, 2, \ldots, n\}$ is a finite subcover.
  - Then $E = \bigcup_{i=1}^{n} U_{p_i}$, and so S is finite.
  - This contradicts our assumption, and so S must have a cluster point.

Corollary

- Every infinite sequence in a compact metric space has a convergent subsequence.

Proof

- Let $\{p_i\}_{i=1}^{\infty}$ be a sequence in a compact metric space E.
- Let $S = \{p_i : i = 1, 2, 3, \ldots\}$.
- Suppose S is finite.
  - Then some point p occurs in the sequence an infinite number of times.
  - Hence $p, p, p, \ldots$ is a convergent subsequence of $\{p_i\}_{i=1}^{\infty}$.
- Suppose S is infinite.
  - Let p be a cluster point of S.
  - For $m = 1, 2, 3, \ldots$, let p_{n_m} be a point in $\{p_i\}_{i=1}^{\infty}$ for which $d(p, p_{n_m}) < \frac{1}{m}$.
  - Given $\epsilon > 0$, choose a positive integer N such that $\frac{1}{N} < \epsilon$.
  - Then $d(p, p_{n_m}) < \epsilon$ for all $m > N$.
  - Hence $\{p_{n_m}\}_{m=1}^{\infty}$ is a convergent subsequence of $\{p_i\}_{i=1}^{\infty}$.

Corollary

- Corollary: A compact metric space is complete.
- Proof:
  - Let $\{p_i\}_{i=1}^{\infty}$ be a Cauchy sequence in a compact metric space E.
  - Then $\{p_i\}_{i=1}^{\infty}$ has a convergent subsequence.
  - Hence $\{p_i\}_{i=1}^{\infty}$ converges, and E is complete.

Corollary

- Corollary: A compact subset of a metric space is closed.
- Proof:
  - Suppose S is a compact subset of a metric space E.
  - Suppose $\{p_i\}_{i=1}^{\infty}$ is a convergent sequence which lies in S.
  - Since S is, as a metric space itself, complete, the limit of $\{p_i\}_{i=1}^{\infty}$ lies in S.
  - Hence S is closed.
- Note: We have now shown that a compact subset of a metric space must be both closed and bounded.