Mathematics 450: Lecture 16

Continuous Functions on Connected Spaces

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Theorem

- Theorem:
  - Suppose E and E' are metric spaces and $f : E \to E'$ is continuous.
  - If f is connected, then $f(E)$ is connected.
- Proof:
  - Suppose $f(E)$ is not connected.
  - Then there exist disjoint nonempty open subsets A and B of $f(E)$ such that $f(E) = A \cup B$.
  - Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nonempty open subsets of E with

$$E = f^{-1}(A) \cup f^{-1}(B).$$

  - This contradicts the assumption that E is connected.
  - Hence $f(E)$ must be connected.

Corollary

- Intermediate value theorem:
  - Suppose $a, b \in \mathbb{R}$ with $a < b$.
  - Suppose $f : [a, b] \to \mathbb{R}$ is continuous.
  - Then, for any real number γ between $f(a)$ and $f(b)$, there exists at least one real number $c \in (a, b)$ such that $f(c) = \gamma$.
- Proof:
  - From the previous theorem, $f([a, b])$ is connected.
  - Since γ is between $f(a)$ and $f(b)$, we have $\gamma \in f([a, b])$.
  - That is, there exists $c \in (a, b)$ for which $f(c) = \gamma$.

Examples

- Example:
  - Suppose $f : [a, b] \to \mathbb{R}$, where $a, b \in \mathbb{R}$ with $a < b$.
  - Then the function $g : [a, b] \to \mathbb{R}^2$ given by $g(x) = (x, f(x))$ is continuous.
  - So $g([a, b])$, that is, the graph of f, is connected.
- More generally, the image of any continuous function $f : [a, b] \to \mathbb{R}^n$ is connected.
- Example:
  - Suppose $p, q \in \mathbb{R}^n$.
  - Define $f : [0, 1] \to \mathbb{R}^n$ by $f(t) = p + t(q - p)$.
  - We call $f([0, 1])$ the *line segment* between p and q.
  - Since f is continuous, the line segment between p and q is connected.

Example

- Let $p = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $q = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$.
- Note: For f as in the previous example, for any $0 \leq t \leq 1$,

$$f(t) = (a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2), \ldots, a_n + t(b_n - a_n)).$$

- Hence

$$d(p, f(t)) = \sqrt{t^2(b_1 - a_1)^2 + t^2(b_2 - a_2)^2 + \cdots + t^2(b_n - a_n)^2} = td(p, q).$$

- Let B be the open ball with center p and radius r.
- Then for any $q \in B$, the line segment from p to q lies entirely within B.
- B is the union of all such line segments, and so is connected by a previous theorem.

Example (cont'd)

- The same argument shows that any closed ball is connected.
- Similarly, $\mathbb{R}^n$ is the union of all line segments starting at the origin, and so is connected.