Mathematics 450: Lecture 18

The Metric Space $C(E)$

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Lemma

- Suppose (E, d) and (E, d') are metric spaces.
- Suppose $f : E \to E'$ and $g : E \to E'$ are continuous.
- For $p \in E$, define $s : E \to \mathbb{R}$ by $s(p) = d'(f(p), g(p))$.
- Then s is continuous on E.

Proof

- Let $p_0 \in E$ and suppose $\epsilon > 0$.
- There exists a δ_1 such that $d'(f(p), f(p_0)) < \frac{\epsilon}{2}$ whenever $d(p, p_0) < \delta_1$.
- And there exists a δ_2 such that $d'(g(p), g(p_0)) < \frac{\epsilon}{2}$ whenever $d(p, p_0) < \delta_2$.
- Let δ be the smaller of δ_1 and δ_2.
- Hence, for $p \in E$ with $d(p, p_0) < \delta$,

$$\begin{aligned}
|s(p) - s(p_0)| &= |d'(f(p), g(p)) - d'(f(p_0), g(p_0))| \\
&= |(d'(f(p), g(p)) - d'(f(p), g(p_0))) + (d'(f(p), g(p_0)) - d'(f(p_0), g(p_0)))| \\
&\leq |d'(f(p), g(p)) - d'(f(p), g(p_0))| + |d'(f(p), g(p_0)) - d'(f(p_0), g(p_0))| \\
&\leq d'(g(p), g(p_0)) + d'(f(p), f(p_0)) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Proposition

- Suppose (E, d) and (E', d') are metric spaces.
- Suppose E is compact and $\mathcal{F}$ is the set of all continuous functions from E to E'.
- For any $f, g \in \mathcal{F}$, let

$$D(f, g) = \max\{d'(f(p), g(p)) \ : \ p \in E\}.$$

- Then $(\mathcal{F}, D)$ is a metric space.

Proof

- Note that D is defined because a continuous function on a compact metric space attains a maximum value.
- Clearly, for all $f, g \in \mathcal{F}$, $D(f,g) \geq 0$, $D(f,g) = 0$ if and only if $f = g$, and $D(f,g) = D(g,f)$.
- Now let $f, g, h \in \mathcal{F}$.
- Then, for some $p_0 \in E$,

$$\begin{aligned} D(f,h) &= d'(f(p_0), h(p_0)) \\ &\leq d'(f(p_0), g(p_0)) + d'(g(p_0), h(p_0)) \\ &\leq \max\{d'(f(p), g(p)) \ : \ p \in E\} + \max\{d'(g(p), h(p)) \ : \ p \in E\} \\ &= D(f,g) + D(g,h). \end{aligned}$$

Convergence

- Let $\mathcal{F}$ be as above.
- Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{F}$.
- Suppose $f \in \mathcal{F}$ with $\lim_{n \to \infty} f_n = f$.
- Then, for every $\epsilon > 0$, there exists an integer N such that, if $n > N$, then

$$D(f_n, f) = \max\{d'(f_n(p), f(p)) \ : \ p \in E\} < \epsilon.$$

- Hence $d'(f_n(p), f(p)) < \epsilon$ for all $p \in E$.
- Hence f_n converges uniformly to f.
- Conversely, if $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f, then $f \in \mathcal{F}$ and $\lim_{n \to \infty} f_n = f$ in $\mathcal{F}$.

Completeness

- Suppose E' is complete and $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{F}$.
- Then f_n converges uniformly to some $f : E \to E'$.
- Moreover, $f \in \mathcal{F}$.
- Hence $\mathcal{F}$ is complete.

Theorem

- Theorem:
  - Let (E, d) and (E', d') be metric spaces.
  - Suppose E is compact and E' is complete.
  - Then the set of all continuous functions from E to E', with metric D, is a complete metric space.
- Notation: We denote the set of all real-valued continuous functions on a compact metric space E by $C(E)$.
- Note: For $f, g \in C(E)$,

$$D(f,g) = \max\{|f(p) - g(p)| \ : \ p \in E\}.$$

Example

- Let $f(x) = 0$ for all $x \in [0,1]$.
- Let B be the closed ball in $C([0,1])$ with center f and radius 1.
- For $n = 1, 2, 3, \ldots$, define

$$f_n(x) = \begin{cases} 1, & \text{if } 0 \le x \le \frac{1}{n+1}, \\ n(n+1)\left(\frac{1}{n} - x\right), & \text{if } \frac{1}{n+1} < x < \frac{1}{n}, \\ 0, & \text{if } \frac{1}{n} \le x \le 1. \end{cases}$$

- Then, for $n = 1, 2, 3, \ldots$, $f_n \in B$.
- And $D(f_n, f_m) = 1$ for all $n, m \in \mathbb{N}$, $n \neq m$.
- Note: It follows that B is not totally bounded.

Example (cont'd)

- Graphs of $y = f_2(x)$ and $y = f_3(x)$: