Mathematics 450: Lecture 18

The Metric Space C(E)

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• Then s is continuous on E.

Proof

- Let $p_0 \in E$ and suppose $\epsilon > 0$.
- There exists a δ_1 such that $d'(f(p), f(p_0)) < \frac{\epsilon}{2}$ whenever $d(p, p_0) < \delta_1$.
- And there exists a δ_2 such that $d'(g(p),g(p_0))<\frac{\epsilon}{2}$ whenever $d(p,p_0)<\delta_2$.
- Let δ be the smaller of δ_1 and δ_2 .
- Hence, for $p \in E$ with $d(p, p_0) < \delta$,

$$|s(p) - s(p_0)| = |d'(f(p), g(p)) - d'(f(p_0), g(p_0))$$

$$= |(d'(f(p), g(p)) - d'(f(p), g(p_0))) + (d'(f(p), g(p_0)) - d'(f(p_0), g(p_0)))|$$

$$\leq |d'(f(p), g(p)) - d'(f(p), g(p_0))| + |d'(f(p), g(p_0)) - d'(f(p_0), g(p_0))|$$

$$\leq d'(g(p), g(p_0)) + d'(f(p), f(p_0))$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon.$$

Proposition

Lemma

• Suppose (E, d) and (E', d') are metric spaces.

• Suppose (E, d) and (E, d') are metric spaces. • Suppose $f: E \to E'$ and $g: E \to E'$ are continuous.

• For $p \in E$, define $s : E \to \mathbb{R}$ by s(p) = d'(f(p), g(p)).

- Suppose E is compact and \mathcal{F} is the set of all continuous functions from E to E'.
- For any $f, g \in \mathcal{F}$, let

$$D(f,g) = \max\{d'(f(p),g(p)) : p \in E\}.$$

• Then (\mathcal{F}, D) is a metric space.

Proof

- Note that D is defined because a continuous function on a compact metric space attains a maximum value.
- Clearly, for all $f, g \in \mathcal{F}$, $D(f, g) \ge 0$, D(f, g) = 0 if and only if f = g, and D(f, g) = D(g, f).
- Now let $f, g, h \in \mathcal{F}$.
- Then, for some $p_0 \in E$,

$$D(f,h) = d'(f(p_0), h(p_0))$$

$$\leq d'(f(p_0), g(p_0)) + d'(g(p_0), h(p_0))$$

$$\leq \max\{d'(f(p), g(p)) : p \in E\} + \max\{d'(g(p), h(p)) : p \in E\}$$

$$= D(f,g) + D(g,h).$$

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Mathematics 450: Lecture 18

O-t-b-- E 2016

Convergence

- Let \mathcal{F} be as above.
- Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence in \mathcal{F} .
- Suppose $f \in \mathcal{F}$ with $\lim_{n \to \infty} f_n = f$.
- Then, for every $\epsilon > 0$, there exists an integer N such that, if n > N, then

$$D(f_n, f) = \max\{d'(f_n(p), f(p)) : p \in E\} < \epsilon.$$

- Hence $d'(f_n(p), f(p)) < \epsilon$ for all $p \in E$.
- Hence f_n converges uniformly to f.
- Conversely, if $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f, then $f \in \mathcal{F}$ and $\lim_{n \to \infty} f_n = f$ in \mathcal{F} .

Completeness

- Suppose E' is complete and $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in \mathcal{F} .
- Then f_n converges uniformly to some $f: E \to E'$.
- Moreover, $f \in \mathcal{F}$.
- Hence F is complete.

Theorem

- Theorem:
 - Let (E, d) and (E', d') be metric spaces.
 - Suppose E is compact and E' is complete.
 - Then the set of all continuous functions from E to E', with metric D, is a complete metric space.
- Notation: We denote the set of all real-valued continuous functions on a compact metric space E by C(E).
- Note: For $f, g \in C(E)$,

$$D(f,g) = \max\{|f(p) - g(p)| : p \in E\}.$$

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Example

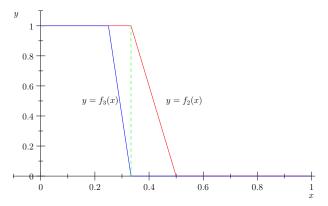
- Let f(x) = 0 for all $x \in [0, 1]$.
- Let B be the closed ball in C([0,1]) with center f and radius 1.
- For n = 1, 2, 3, ..., define

$$f_n(x) = \begin{cases} 1, & \text{if } 0 \le x \le \frac{1}{n+1}, \\ n(n+1)\left(\frac{1}{n} - x\right), & \text{if } \frac{1}{n+1} < x < \frac{1}{n}, \\ 0, & \text{if } \frac{1}{n} \le x \le 1. \end{cases}$$

- Then, for $n = 1, 2, 3, ..., f_n \in B$.
- And $D(f_n, f_m) = 1$ for all $n, m \in \mathbb{N}$, $n \neq m$.
- Note: It follows that *B* is not totally bounded.

Example (cont'd)

• Graphs of $y = f_2(x)$ and $y = f_3(x)$:



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October 5, 2016

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10