Mathematics 450: Lecture 19

The Derivative

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October 10, 2016

Definition

- Suppose f is a real-valued function defined on an open set $U \subset \mathbb{R}$.
- Let $x_0 \in U$.
- We say f is *differentiable at* x_0 if
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
exists.
- Notation: If f is differentiable at x_0, we write
$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$
- Equivalently, we have
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Proposition

- Definition: We say a function $\varphi : \mathbb{R} \to \mathbb{R}$ is *linear* if there are constants c and k for which $\varphi(x) = cx + k$ for all $x \in \mathbb{R}$.
- Proposition:
  - Suppose f is a real-valued function defined on an open set $U \subset \mathbb{R}$.
  - Then f is differentiable at x_0 if and only if there exists a linear function $\varphi : \mathbb{R} \to \mathbb{R}$ such that for any given $\epsilon > 0$ there exists $\delta > 0$ such that
  $$|f(x) - \varphi(x)| \le \epsilon |x - x_0|$$
  whenever $|x - x_0| < \delta$.

Proof

- Suppose f is differentiable at x_0.
  - Given any $\epsilon > 0$, there exists a $\delta > 0$ such that, if $|x - x_0| < \delta$ and $x \neq x_0$,
  $$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \le \epsilon.$$
  - Equivalently,
  $$|f(x) - (f'(x_0)(x - x_0) + f(x_0))| \le \epsilon |x - x_0|.$$
  - Note that this last inequality holds for $x = x_0$ as well.
  - Hence, letting $\varphi(x) = f'(x_0)(x - x_0) + f(x_0)$, we have
  $$|f(x) - \varphi(x)| \le \epsilon |x - x_0|$$
  whenever $|x - x_0| < \delta$.

Proof (cont'd)

- Suppose there exists a linear function $\varphi : \mathbb{R} \to \mathbb{R}$ such that for any given $\epsilon > 0$ there exists $\delta > 0$ such that
$$|f(x) - \varphi(x)| \leq \epsilon |x - x_0|$$
whenever $|x - x_0| < \delta$.
    - Given $\epsilon > 0$, choose $\delta > 0$ such that $|f(x) - \varphi(x)| \leq \epsilon|x - x_0|$ whenever $|x - x_0| < \delta$.
    - Then $|f(x_0) - \varphi(x_0)| = 0$, so $\varphi(x_0) = f(x_0)$.
    - So if $\varphi(x) = cx + k$ for some $c, k \in \mathbb{R}$, then $f(x_0) = \varphi(x_0) = cx_0 + k$.
    - Hence $\varphi(x) = cx + (f(x_0) - cx_0) = c(x - x_0) + f(x_0)$.
    - Thus
    $$|f(x) - f(x_0) - c(x - x_0)| \leq \epsilon |x - x_0|$$
    whenever $|x - x_0| < \delta$.
    - That is, if $x \neq x_0$ and $|x - x_0| < \delta$, then
    $$\left| \frac{f(x) - f(x_0)}{x - x_0} - c \right| \leq \epsilon.$$

Proof (cont'd)

- That is,
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = c.$$
- That is, f is differentiable at x_0 and $f'(x_0) = c$.

Proposition

- Suppose $U \subset \mathbb{R}$ is open and $f : U \to \mathbb{R}$.
- If f is differentiable at $x_0 \in U$, then f is continuous at x_0.

Proof

- Suppose f is differentiable at x_0.
- Choose a real number $\delta_1 > 0$ such that, if $|x - x_0| < \delta_1$,
$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq |x - x_0|.$$
- Then
$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f(x_0) - f'(x_0)(x - x_0) + f'(x_0)(x - x_0)| \\ &\leq |f(x) - f(x_0) - f'(x_0)(x - x_0)| + |f'(x_0)||x - x_0| \\ &\leq (1 + |f'(x_0)|)|x - x_0|. \end{aligned}$$
- Given $\epsilon > 0$, let δ be the smaller of δ_1 and $\frac{\epsilon}{1 + |f'(x_0)|}$.
- Then, if $|x - x_0| < \delta$,
$$|f(x) - f(x_0)| \leq (1 + |f'(x_0)|)|x - x_0| < \epsilon.$$

Example

- Let $f(x) = |x|$.
- For $x \neq 0$,

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} -1, & \text{if } x < 0, \\ 1, & \text{if } x > 0. \end{cases}$$

- Hence $\frac{f(x)-f(0)}{x-0}$ does not have a limit as x approaches 0.
- Thus f is not differentiable at 0.
- However, f is continuous at 0.

Definition

- Suppose $U \subset \mathbb{R}$ is open and $f : U \to \mathbb{R}$.
- We say f is *differentiable* on U if f is differentiable at x for all $x \in U$.
- Note:
  - If f is differentiable on U, we let f' denote the real-valued function defined by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

    for all $x_0 \in U$.
  - Alternative notations for f' include $\dfrac{df}{dx}$ and $\dfrac{df(x)}{dx}$.

Example

- Let $m, b \in \mathbb{R}$.
- Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = mx + b$.
- Then, for any $x_0 \in \mathbb{R}$,

$$\begin{aligned} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{(mx + b) - (mx_0 + b)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{m(x - x_0)}{x - x_0} \\ &= m. \end{aligned}$$

- Hence f is differentiable on $\mathbb{R}$ and $f'(x) = m$ for all $x \in \mathbb{R}$.
- In particular,

$$\frac{dx}{dx} = 1 \text{ and } \frac{dc}{dx} = 0 \text{ for any } c \in \mathbb{R}.$$