Mathematics 450: Lecture 29

Properties of Exponential Functions

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Properties of the exponential function

- Proposition: For $x, y \in \mathbb{R}$,
$$\exp(x+y) = \exp(x)\exp(y).$$
- Proof:
  - We have
$$\log(\exp(x)\exp(y)) = \log(\exp(x)) + \log(\exp(y)) = x + y.$$
  - So
$$\exp(x+y) = \exp(x)\exp(y).$$

Properties (cont'd)

- Proposition: For $x, y \in \mathbb{R}$,
$$\exp(x-y) = \frac{\exp(x)}{\exp(y)}.$$
- Proof:
  - We have
$$\log\left(\frac{\exp(x)}{\exp(y)}\right) = \log(\exp(x)) - \log(\exp(y)) = x - y.$$
  - So
$$\exp(x-y) = \frac{\exp(x)}{\exp(y)}.$$
- Note: It follows that
$$\exp(-x) = \frac{\exp(0)}{\exp(x)} = \frac{1}{\exp(x)}.$$

Properties (cont'd)

- Proposition: For any $x \in \mathbb{R}$ and integer n,
$$\exp(nx) = (\exp(x))^n.$$
- Proof:
  - We have
$$\log((\exp(x))^n) = n\log(\exp(x)) = nx.$$
  - So
$$\exp(nx) = (\exp(x))^n.$$

Definition

- Definition: If $x, n \in \mathbb{R}$ with $x > 0$, we define
$$x^n = \exp(n \log(x)).$$
- Note:
  - If n is a positive integer, then we already knew that
$$\exp(n \log(x)) = \exp\left(\log\left(x^n\right)\right) = x^n.$$
  - So the definition is consistent with our prior definition of x^n for integers n.

Properties

- Proposition: For any $x, y, n, m \in \mathbb{R}$ with $x, y > 0$:
  - $x^n x^m = x^{n+m}$,
  - $\frac{x^n}{x^m} = x^{n-m}$,
  - $(x^n)^m = x^{nm}$,
  - $(xy)^n = x^n y^n$.
- Proof:
  - These all follow easily from the definition and the properties of the exponential function.
  - For example,
$$x^n x^m = \exp(n \log(x)) \exp(m \log(x)) = \exp((n+m) \log(x)) = x^{n+m}.$$

Properties (cont'd)

- Note:
  - If $x > 0$ and n is a positive integer, then
$$(x^{\frac{1}{n}})^n = x^{n \cdot \frac{1}{n}} = x.$$
  - That is, $x^{\frac{1}{n}}$ is the nth root of x.
  - For any positive real number n, we let $0^n = 0$.

Proposition

- Proposition: If $x, n \in \mathbb{R}$ with $x > 0$, then
$$\frac{d}{dx} x^n = n x^{n-1}.$$
- Proof:
$$\frac{d}{dx} x^n = \frac{d}{dx} \exp(n \log(x)) = \exp(n \log(x)) \cdot \frac{n}{x} = \frac{n x^n}{x} = n x^{n-1}.$$

e

- Definition: $e = \exp(1)$.
- Note: For any real number x,

$$e^x = \exp(x \log(e)) = \exp(x).$$

- We have seen that $\frac{1}{2} < \log(2) < 1$.
- It follows that $1 < \log(4) < 2$.
- Hence $\log(2) < 1 < \log(4)$.
- It follows that $2 < e < 4$.