Mathematics 450: Lecture 30

Interchange of Limit Operations

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Example

- Note: If $a > 0$, then
$$\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\log(a)} = e^0 = 1.$$
- Define $f : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ by $f(m,n) = \left(\frac{1}{m}\right)^{\frac{1}{n}}$.
- Then
$$\lim_{n\to\infty}\left(\lim_{m\to\infty} f(n,m)\right) = \lim_{n\to\infty} 0 = 0,$$
but
$$\lim_{m\to\infty}\left(\lim_{n\to\infty} f(n,m)\right) = \lim_{m\to\infty} 1 = 1,$$

Example

- Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions which converge uniformly to a function f on $[0,1]$.
- Note: It follows that f is also continuous.
- Then, for any $x_0 \in [0,1]$,
$$\lim_{x\to x_0}\left(\left(\lim_{n\to\infty} f_n\right)(x)\right) = \lim_{x\to x_0} f(x) = f(x_0)$$
and
$$\lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right) = \lim_{n\to\infty} f_n(x_0) = f(x_0).$$

Example

- For $n = 1, 2, 3, \ldots$, define $f_n : [0,1] \to \mathbb{R}$ by
$$f_n(x) = \begin{cases} 4n^2 x, & \text{if } 0 \le x \le \frac{1}{2n}, \\ 4n - 4n^2 x, & \text{if } \frac{1}{2n} < x \le \frac{1}{n}, \\ 0, & \text{if } \frac{1}{n} < x \le 1. \end{cases}$$
- Let $f(x) = 0$ for all $x \in [0,1]$.
- Then $\lim_{n\to\infty} f_n = f$.
- But
$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty} 1 = 1,$$
while
$$\int_0^1 \lim_{n\to\infty} f_n(x)dx = \int_0^1 f(x)dx = 0.$$

Theorem

- Suppose $a, b \in \mathbb{R}$, $a < b$.
- Suppose, for $n = 1, 2, 3, \ldots$, $f_n : [a, b] \to \mathbb{R}$ is continuous.
- Suppose $\{f_n\}_{n=1}^{\infty}$ converges uniformly and let $f = \lim_{n\to\infty} f_n$.
- Then
$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof

- Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $x \in [a, b]$,
$$|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$$
for all $n > N$.
- Hence, for all $n > N$ and $x \in [a, b]$,
$$-\frac{\epsilon}{b-a} < f_n(x) - f(x) < \frac{\epsilon}{b-a}.$$
- Hence, for all $n > N$,
$$-\epsilon \leq \int_a^b (f_n(x) - f(x))dx \leq \epsilon.$$
- That is, for all $n > N$,
$$\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| \leq \epsilon.$$

Proof (cont'd)

- Thus
$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Theorem

- Suppose $a, b \in \mathbb{R}$, $a < b$.
- Suppose, for $n = 1, 2, 3, \ldots$, $f_n : [a, b] \to \mathbb{R}$ is integrable.
- Suppose $\{f_n\}_{n=1}^{\infty}$ converges uniformly and let $f = \lim_{n\to\infty} f_n$.
- Then
$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof

- Note: The proof is the same as the previous theorem if we can show that f is integrable.
- Given $\epsilon > 0$, there exists a positive integer n such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3(b-a)}$$

  for all $x \in [a, b]$.
- Moreover, since f_n is integrable, there exist step functions g_1 and g_2 on $[a, b]$ such that $g_1(x) \le f_n(x) \le g_2(x)$ for all $x \in [a, b]$ and

$$\int_a^b (g_2(x) - g_1(x))dx < \frac{\epsilon}{3}.$$

Proof (cont'd)

- Now, for all $x \in [a, b]$,

$$f_n(x) - \frac{\epsilon}{3(b-a)} < f(x) < f_n(x) + \frac{\epsilon}{3(b-a)}.$$

- So, for all $x \in [a, b]$,

$$g_1(x) - \frac{\epsilon}{3(b-a)} < f(x) < g_2(x) + \frac{\epsilon}{3(b-a)}.$$

- Let

$$h_1(x) = g_1(x) - \frac{\epsilon}{3(b-a)} \text{ and } h_2(x) = g_2(x) + \frac{\epsilon}{3(b-a)}.$$

- Then h_1 and h_2 are step functions with $h_1(x) \le f(x) \le g_2(x)$ for all $x \in [a, b]$.

Proof (cont'd)

- And

$$\int_a^b (h_2(x) - h_1(x))dx = \int_a^b (g_2(x) - g_1(x))dx + \frac{2\epsilon}{3(b-a)} \int_a^b dx < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.$$

- Hence f is integrable and the result follows.

Theorem

- Suppose $U \subset \mathbb{R}$ is an open interval.
- For $n = 1, 2, 3, \ldots$, suppose $f_n : U \to \mathbb{R}$ has a continuous derivative f_n'.
- Suppose the sequence $\{f_n'\}_{n=1}^{\infty}$ converges uniformly on U and let $g = \lim_{n\to\infty} f_n'$.
- Suppose for some $a \in U$ the sequence $\{f_n(a)\}_{n=1}^{\infty}$ converges.
- Then $\{f_n\}_{n=1}^{\infty}$ converges, $f = \lim_{n\to\infty} f_n$ is differentiable, and

$$f' = \lim_{n\to\infty} f_n'.$$

Proof

- We know that, for any $x \in U$ and $n \in \mathbb{N}$,

$$\int_a^x f_n'(t)dt = f_n(x) - f_n(a).$$

- Hence, for any $x \in U$,

$$\lim_{n\to\infty}(f_n(x) - f_n(a)) = \lim_{n\to\infty}\int_a^x f_n'(t)dt = \int_a^x g(t)dt.$$

- Hence, for any $x \in U$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\int_a^x f_n'(t)dt + \lim_{n\to\infty} f_n(a) = \int_a^x g(t)dt + f(a).$$

- Let $f = \lim_{n\to\infty} f_n$.

Proof (cont'd)

- Hence we have

$$f(x) = f(a) + \int_a^x g(t)dt$$

  for all $x \in U$.
- Hence $f' = g$.