# Mathematics 450: Lecture 34

## Series of Functions

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## Proposition

- Suppose $E$ is a metric space.
- Suppose, for $n = 1, 2, 3, \ldots$, $f_n : E \to \mathbb{R}$.
- Then $\sum_{n=1}^{\infty} f_n$ converges uniformly if and only if, given any $\epsilon > 0$, there exists a positive integer $N$ such that, if $n > m \geq N$, then

$$|f_{m+1}(p) + f_{m+2}(p) + \cdots + f_n(p)| < \epsilon$$

  for all $p \in E$.



## Definition

- Suppose $E$ is a metric space.
- Suppose, for $n = 1, 2, 3, \ldots$, $f_n : E \to \mathbb{R}$.
- We say the infinite series $\sum_{n=1}^{\infty} f_n$ *converges absolutely* if the infinite series $\sum_{n=1}^{\infty} f_n(p)$ converges absolutely for each $p \in E$.



## Proposition

- Suppose $E$ is a metric space.
- Suppose, for $n = 1, 2, 3, \ldots$, $f_n : E \to \mathbb{R}$.
- Suppose $\sum_{n=1}^{\infty} a_n$ converges and, for $n = 1, 2, 3, \ldots$, $|f_n(p)| \leq a_n$ for all $p \in E$.
- Then $\sum_{n=1}^{\infty} f_n$ converges absolutely and uniformly.

## Proposition

- Suppose $E$ is a metric space.
- Suppose, for $n = 1, 2, 3, \ldots$, $f_n : E \to \mathbb{R}$ is continuous.
- If $\sum_{n=1}^{\infty} f_n$ converges uniformly, then $f = \sum_{n=1}^{\infty} f_n$ is continuous on $E$.



## Proposition

- Suppose $a, b \in \mathbb{R}$ with $a < b$.
- Suppose, for $n = 1, 2, 3, \ldots$, $f_n : [a, b] \to \mathbb{R}$ is continuous.
- Suppose $\sum_{n=1}^{\infty} f_n$ converges uniformly.
- If $f = \sum_{n=1}^{\infty} f_n$, then

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx.$$



## Proposition

- Suppose $U \subset \mathbb{R}$ is an open interval.
- Suppose, for $n = 1, 2, 3, \ldots$, $f_n : U \to \mathbb{R}$ is differentiable and $f_n'$ is continuous.
- Suppose $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $U$.
- Suppose for some $a \in U$, $\sum_{n=1}^{\infty} f_n(a)$ converges.
- Then
  - $\sum_{n=1}^{\infty} f_n$ converges, and
  - if $f = \sum_{n=1}^{\infty} f_n$, then $f$ is differentiable on $U$ and

$$f' = \sum_{n=1}^{\infty} f_n'.$$



## Definition

- Suppose $a$, $c_0$, $c_1$, $c_2$, ... are real numbers. We call the series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

a *power series*.

## Theorem

- For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, exactly one of the following holds:
  - The series converges absolutely for all $x \in \mathbb{R}$. Moreover, for any positive real number $r_1$, the convergence is uniform on $[a-r_1, a+r_1]$.
  - There exists a positive real number $r$ such that the series converges absolutely for all $x \in \mathbb{R}$ with $|x-a| < r$ and diverges for all $x \in \mathbb{R}$ with $|x-a| > r$. Moreover, for any positive real number $r_1$ with $r_1 < r$, the convergence is uniform on $[a-r_1, a+r_1]$.
  - The series converges only for $x = a$.



## Proof

- Suppose the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $x = \xi$, where $\xi \neq a$.
- Let $b$ be a real number with $0 < b < |\xi - a|$.
- Since $\sum_{n=0}^{\infty} c_n(\xi-a)^n$ converges, $\lim_{n\to\infty} c_n(\xi-a)^n = 0$.
- Hence there exists a real number $M$ for which $|c_n(\xi-a)^n| \leq M$ for $n = 0, 1, 2, 3, \ldots$.
- Now, if $|x-a| \leq b$, for $n = 0, 1, 2, \ldots$,

$$|c_n(x-a)^n| = |c_n(x-a)^n| \cdot \frac{|\xi-a|^n}{|\xi-a|^n} = |c_n(\xi-a)^n| \cdot \left|\frac{x-a}{\xi-a}\right|^n \leq M\left|\frac{b}{\xi-a}\right|^n.$$

- Hence $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely and uniformly on $[a-b, a+b]$ by comparison with the series $\sum_{n=0}^{\infty} M\left|\frac{b}{\xi-a}\right|^n$.



## Proof (cont'd)

- Now let $S = \left\{\xi \in \mathbb{R} \;:\; \sum_{n=0}^{\infty} c_n(\xi-a)^n \text{ converges}\right\}$.
- Then we may have
  - $S = \{a\}$,
  - $S$ is unbounded, in which case $S = \mathbb{R}$, or
  - $S$ is bounded.
- In the last case, let $r = \sup S - a$, and the result follows.



## Definition

- Consider a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$.
- If $r$ is a positive real number for which the series converges for all $x$ with $|x-a| < r$ and diverges for all $x$ with $|x-a| > r$, then we call $r$ the *radius of convergence* of the power series and we call the interval $(a-r, a+r)$ the *interval of convergence*.
- If the series converges for all $x \in \mathbb{R}$, we say the *radius of convergence* is $\infty$ and the *interval of convergence* is $(-\infty, \infty)$.
- If the series converges only if $x = a$, we say the *radius of convergence* is 0.

## Examples

- Example: $\sum_{n=0}^{\infty} x^n$ has radius of convergence 1 and interval of convergence $(-1, 1)$.
- Example:
  - Consider $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
  - Then, for any $x \in \mathbb{R}$, $x \neq 0$,

$$\lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

  - So, by the ratio test, the radius of convergence is $\infty$ and the interval of convergence is $(-\infty, \infty)$.
- Example:
  - Consider $\sum_{n=0}^{\infty} n!x^n$.
  - Then, for any $x \in \mathbb{R}$, $x \neq 0$, $\frac{|(n+1)!x^{n+1}|}{|n!x^n|} = (n+1)|x|$.
  - Hence, by the ratio test, the radius of convergence is 0.