Mathematics 450: Lecture 8

Sequences of Real Numbers

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Sums

- Proposition:
  - Suppose $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ are convergent sequences of real numbers with

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$

  - Then $\{a_i + b_i\}_{i=1}^{\infty}$ is a convergent sequence with

$$\lim_{n\to\infty}(a_n + b_n) = a + b.$$

Proof

- Given $\epsilon > 0$, there exist positive integers N_1 and N_2 such that

$$|a_n - a| < \frac{\epsilon}{2}$$

when $n > N_1$ and

$$|b_n - b| < \frac{\epsilon}{2}$$

when $n > N_2$.
- Let N be the maximum of N_1 and N_2.
- Then, for $n > N$,

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

- Hence $\lim_{n\to\infty}(a_n + b_n) = a + b$.

Products

- Proposition:
  - Suppose $\{a_i\}_{i=1}^{\infty}$ and $\{a_i\}_{i=1}^{\infty}$ are convergent sequences of real numbers with

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$

  - Then $\{a_i b_i\}_{i=1}^{\infty}$ is a convergent sequence with

$$\lim_{n\to\infty} a_n b_n = ab.$$

Proof

- Since convergent sequences are bounded, we may find real numbers M_1 and M_2 such that $|a_n| < M_1$ and $|b_n| < M_2$ for all $n \in \mathbb{N}$.
- Let M be the maximum of M_1 and M_2.
- Then $M > 0$, $|a| \leq M$, and $|b| \leq M$.
- Given $\epsilon > 0$, there exist integers N_1 and N_2 such that

$$|a_n - a| < \frac{\epsilon}{2M}$$

when $n > N_1$ and

$$|b_n - b| < \frac{\epsilon}{2M}$$

when $n > N_2$.
- Let N be the maximum of N_1 and N_2.

Proof (cont'd)

- Then, for any $n > N$,

$$\begin{aligned} |a_b b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &= |b_n(a_n - a) + a(b_n - b)| \\ &\leq |b_n(a_n - a)| + |a(b_n - b)| \\ &= |b_n||a_n - a| + |a||b_n - b| \\ &< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\ &= \epsilon. \end{aligned}$$

- Hence $\lim_{n\to\infty} a_n b_n = ab$.

Reciprocals

- Proposition:
  - Suppose $\{a_i\}_{i=1}^{\infty}$ is a convergent sequence of real numbers with $\lim_{n\to\infty} a_n = a$.
  - Suppose $a \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$.
  - Then $\{\frac{1}{a_i}\}_{i=1}^{\infty}$ is a convergent sequence with

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}.$$

Proof

- Since $|a| > 0$, we may find a positive integer N_1 such that

$$|a_n - a| < \frac{|a|}{2}$$

for all $n > N_1$.
- Then, for $n > N_1$,

$$|a_n| = |a - (a - a_n)| \geq ||a| - |a - a_n|| = |a| - |a - a_n| > |a| - \frac{|a|}{2} = \frac{|a|}{2}.$$

- Given $\epsilon > 0$, there exists a positive integer N_2 such that

$$|a_n - a| < \frac{|a|^2\epsilon}{2}$$

for all $n > N_2$.
- Let N be the larger of N_1 and N_2.

Proof (cont'd)

- We now have, for $n > N$,

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a - a_n|}{|a_n a|} < \frac{\frac{|a|^2\epsilon}{2}}{|a| \cdot \frac{|a|}{2}} = \epsilon.$$

- Thus

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}.$$

Constant multiples

- Proposition:
  - Suppose $\{a_i\}_{i=1}^{\infty}$ is a convergent sequence of real numbers with

$$\lim_{n\to\infty} a_n = a.$$

  - Then, for any real number c, $\{ca_i\}_{i=1}^{\infty}$ is a convergent sequence with

$$\lim_{n\to\infty} ca_n = ca.$$

- Proof: Apply the result for products with $b_i = c$ for $i = 1, 2, 3, \ldots$.
- Note: In particular, if $c = -1$, we have

$$\lim_{n\to\infty} (-a_n) = -a.$$

Differences

- Proposition:
  - Suppose $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ are convergent sequences of real numbers with

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$

  - Then $\{a_i - b_i\}_{i=1}^{\infty}$ is a convergent sequence with

$$\lim_{n\to\infty} (a_n - b_n) = a - b.$$

- Proof: Apply the results for sums with the second sequence being $\{-b_i\}_{i=1}^{\infty}$.

Quotients

- Proposition:
  - Suppose $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ are convergent sequences of real numbers with

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$

  - Suppose $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$.
  - Then $\{\frac{a_i}{b_i}\}_{i=1}^{\infty}$ is a convergent sequence with

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

  - Proof: Apply the result for products with the second sequence being $\left\{\frac{1}{b_i}\right\}_{i=1}^{\infty}$.

Proposition

- Suppose $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ are convergent sequences of real numbers with
$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$
- Suppose $a_n \leq b_n$ for $n = 1, 2, 3, \ldots$.
- Then $a \leq b$.
- Proof:
  - Since $b_n - a_n \geq 0$ for all n, and $\{x : x \geq 0\}$ is a closed set,
$$b - a = \lim_{n\to\infty}(b_n - a_n) \geq 0.$$
  - Hence $b \geq a$.

Monotonic sequences

- Definition: We say a sequence of real numbers $\{a_i\}_{i=1}^{\infty}$ is *increasing* if $a_1 \leq a_2 \leq a_3 \leq \cdots$.
- Definition: We say a sequence of real numbers $\{a_i\}_{i=1}^{\infty}$ is *decreasing* if $a_1 \geq a_2 \geq a_3 \geq \cdots$.
- Definition: We say a sequence of real numbers $\{a_i\}_{i=1}^{\infty}$ is *monotonic* if it is either increasing or decreasing.

Theorem

- Theorem: A bounded monotonic sequence of real numbers is convergent.
- Proof:
  - Suppose $\{a_i\}_{i=1}^{\infty}$ is a bounded increasing sequence of real numbers.
  - Let $a = \sup\{a_1, a_2, \ldots\}$.
  - Note: $a_n \leq a$ for all n.
  - Given $\epsilon > 0$, there exists a positive integer N for which $a - \epsilon < a_N$.
  - Since $\{a_i\}_{i=1}^{\infty}$ is increasing, it follows that
$$a - \epsilon < a_n < a + \epsilon$$
for all $n > N$.
  - Hence $|a_n - a| < \epsilon$ for all $n > N$, and so $\lim_{n\to\infty} a_n = a$.
  - The proof for decreasing sequences is similar.

Example

- Suppose $0 \leq a < 1$.
- Then $\{a^n\}_{n=1}^{\infty}$ is a bounded decreasing sequence.
- Let $L = \lim_{n\to\infty} a^n$.
- Then
$$aL = a \lim_{n\to\infty} a^n = \lim_{n\to\infty} a^{n+1} = L.$$
- Hence $0 = L - aL = L(1 - a)$, and so $L = 0$ since $1 - a \neq 0$.
- Note:
  - Since $|a^n| = |a|^n$, the result extends to show that
$$\lim_{n\to\infty} a^n = 0 \text{ whenever } -1 < a < 1.$$
  - This also shows that if a is a real number with $|a| > 1$, then the sequence $\{a^n\}_{n=1}^{\infty}$ is unbounded.