Mathematics 160: Lecture 10

Determinants

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Definition

- If $A = [a]$ is a 1×1 matrix, then $\det A = a$.
- If $A = [a_{ij}]$ is a 2×2 matrix, then the *determinant* of A is
$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$
- Example:
$$\det \begin{bmatrix} 1 & 2 \\ 5 & -4 \end{bmatrix} = (1)(-4) - (2)(5) = -14.$$

Definition

- If $A = [a_{ij}]$ is a 3×3 matrix, then the *determinant* of A is
$$\det A = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Example

- We have
$$\begin{aligned} \det \begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 3 \\ -1 & 2 & 5 \end{bmatrix} &= 3 \det \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix} - 2 \begin{bmatrix} 4 & 3 \\ -1 & 5 \end{bmatrix} - 1 \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix} \\ &= 3(10 - 6) - 2(20 + 3) - (8 + 2) \\ &= 12 - 46 - 10 \\ &= -44. \end{aligned}$$

Definition

- For an $n \times n$ matrix $A = [a_{ij}]$, let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column. We call

$$C_{ij}(A) = (-1)^{i+j} \det A_{ij}$$

the (i,j)-*cofactor* of A.
- For an $n \times n$ matrix $A = [a_{ij}]$, the *determinant* of A is

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Example

- We have

$$\det \begin{bmatrix} 4 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & -2 & 0 & 4 \\ 1 & -1 & 1 & 5 \end{bmatrix} = 4 \det \begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 4 \\ -1 & 1 & 5 \end{bmatrix} - 2 \det \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$
$$+ 3 \det \begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 4 \\ 1 & -1 & 5 \end{bmatrix} - 2 \det \begin{bmatrix} 2 & 1 & 0 \\ 3 & -2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$
$$= 4(-4-0-2) - 2(-8+3) + 3(2(-10+4)$$
$$-(15-4) + (-3+2)) - 2(-4-3+0)$$
$$= -24 + 10 - 72 + 14$$
$$= -72.$$

Theorem

- If $A = [a_{ij}]$ is an $n \times n$ matrix, then the cofactor expansions along any row or column are all equal.
- That is, for any fixed i,

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in},$$

and for any fixed j,

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Example

- For our previous example,

$$\det \begin{bmatrix} 4 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & -2 & 0 & 4 \\ 1 & -1 & 1 & 5 \end{bmatrix} = 3 \det \begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 4 \\ 1 & -1 & 5 \end{bmatrix} - \det \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & -2 & 4 \end{bmatrix}$$
$$= 3(2(-10+4) - (15-4) + (-3+2))$$
$$-(4(4+2) - 2(8-3) + 2(-4-3))$$
$$= 3(-12-11-1) - (24-10-14)$$
$$= -72.$$