Mathematics 160: Lecture 11

Determinants: Properties

Dan Sloughter

Furman University

September 23, 2011

Theorem

- Suppose A is an $n \times n$ matrix. If B is obtained from A by exchanging two rows, or two columns, then $\det B = -\det A$.
- Reason:
  - The theorem is true for $n = 1$ and $n = 2$.
  - The rest of the proof follows from induction:
    - Assume the result is true for $(n-1) \times (n-1)$ matrices and let A be an $n \times n$ matrix.
    - Compute $\det B$ by expanding along a row, or column, other than the two that were switched.
    - Since the cofactors in this expansion are the negative of the cofactors in the expansion for $\det A$ along the same row or column, it follows that $\det B = -\det A$.

Theorem

- If two rows (or two columns) of an $n \times n$ matrix A are equal, then $\det A = 0$.
- Reason:
  - Let B be the matrix obtained by exchanging the two rows (or columns) which are equal.
  - Then $B = A$, so $\det B = \det A$.
  - But we also have $\det B = -\det A$.
  - Hence $\det A = -\det A$, implying that $\det A = 0$.

Theorem

- Suppose A is an $n \times n$ matrix. If B is obtained from A by multiplying a row, or column, of A by a scalar k, then $\det B = k \det A$.
- Reason: If we expand $\det B$ along the row, or column, multiplied by k, every term in the expansion is multiplied by k.

Theorem

- Suppose A is an $n \times n$ matrix. If B is obtained from A by adding a multiple of a row (or column) of A to another row (or column), then $\det B = \det A$.
- Reason:
  - Suppose, for example, that B is obtained from A by adding k times the first row to the second row.
  - Expanding $\det B$ along the second row, we have

$$\begin{aligned}\det B &= (a_{21} + ka_{11})C_{21} + (a_{22} + ka_{12})C_{22} + (a_{23} + ka_{13})C_{23} \\ &\quad + \cdots + (a_{2n} + ka_{1n})C_{2n} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + \cdots + a_{2n}C_{2n} \\ &\quad + k(a_{11}C_{21} + a_{12}C_{22} + a_{13}C23 + \cdots + a_{1n}C_{2n}) \\ &= \det A + k \det C,\end{aligned}$$

    where C is the matrix obtained from A by replacing the second row by the first row.
  - But then $\det C = 0$, so $\det B = \det A$.

Example

- We have

$$\det \begin{bmatrix} 1 & -5 & 3 \\ 5 & 16 & 4 \\ -3 & 15 & -9 \end{bmatrix} = \det \begin{bmatrix} 1 & -5 & 3 \\ 5 & 16 & 4 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Example

- We have

$$\begin{aligned}\det \begin{bmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{bmatrix} &= -\det \begin{bmatrix} 3 & 0 & -3 & 6 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{bmatrix} \\ &= -3\det \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{bmatrix}\end{aligned}$$

Example (cont'd)

- Continuing,

$$\begin{aligned}&= -3\det \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 0 & 4 & 7 & 3 \\ 0 & -1 & 2 & -9 \end{bmatrix} = 3\det \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -9 \\ 0 & 4 & 7 & 3 \\ 0 & 2 & -4 & 5 \end{bmatrix} \\ &= -3\det \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 9 \\ 0 & 4 & 7 & 3 \\ 0 & 2 & -4 & 5 \end{bmatrix} = -3\det \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 9 \\ 0 & 0 & 15 & -33 \\ 0 & 0 & 0 & -13 \end{bmatrix} \\ &= (-3)(1)(1)(15)(-13) = 585.\end{aligned}$$

Theorem

- The following results follow immediately from our previous work:
  - If A is an $n \times n$ matrix with a row, or column, of zeros, then $\det A = 0$.
  - If A is an $n \times n$ matrix and k is a scalar, then $\det kA = k^n \det A$.
  - If A is an $n \times n$ matrix, $\det A^T = \det A$.

Upper triangular matrices

- We call an $n \times n$ matrix $A = [a_{ij}]$ *upper triangular* if $a_{ij} = 0$ for all $i > j$.
- In other words, A is upper triangular if all entries below the main diagonal are 0.
- Theorem: If A is upper triangular, then

$$\det A = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

Elementary matrices

- Suppose E is an elementary matrix obtained from I_n by switching two rows. Then
$$\det E = -\det I_n = -1.$$
- Suppose E is an elementary matrix obtained from I_n by adding a constant multiple of one row to another. Then
$$\det E = \det I_n = 1.$$
- Suppose E is an elementary matrix obtained from I_n by multiplying a row by a scalar k. Then
$$\det E = k \det I_n = k.$$

Multiplication by elementary matrices

- Let B be an $n \times n$ matrix.
- Suppose E is an elementary matrix obtained from I_n by switching two rows. Then
$$\det EB = -\det B = \det E \det B.$$
- Suppose E is an elementary matrix obtained from I_n by adding a constant multiple of one row to another. Then
$$\det EB = \det B = \det E \det B.$$
- Suppose E is an elementary matrix obtained from I_n by multiplying a row by a scalar k. Then
$$\det EB = k \det B = \det E \det B.$$
- Hence: if B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then
$$\det EB = (\det E)(\det B).$$

Theorem

- An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.
- Reason:
  - First suppose A is invertible.
  - Then $A = E_1 E_2 \cdots E_k$ for some elementary matrices E_1, E_2, ..., E_k.
  - Hence $\det A = (\det E_1)(\det E_2) \cdots (\det E_k) \neq 0$.
  - Now suppose $\det A \neq 0$.
  - Let R be the reduced row-echelon form of A and let E_1, E_2, ..., E_k be elementary matrices for which $R = E_k E_{k-1} \cdots E_1 A$.
  - Then
    $$\det R = (\det E_k)(\det E_{k-1}) \cdots (\det E_1)(\det A) \neq 0.$$
  - Hence R is an $n \times n$ matrix in reduced row-echelon form without a row of zeros.
  - Hence $R = I_n$, and A is invertible.