Mathematics 160: Lecture 12

Adjoints

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Theorem

- If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B).$$

- Reason:
  - Note: if A is not invertible, then neither is AB (since, if AB were invertible with inverse C, then BC would be an inverse for A).
  - So if A is not invertible, we have $\det A = 0$ and $\det AB = 0$, and so

$$\det AB = 0 = (\det A)(\det B).$$

  - If A is invertible, then $A = E_1 E_2 \cdots E_k$ for some elementary matrices $E_1, E_2, \ldots, E_k$.
  - Then, using our results about products of elementary matrices,

$$\begin{aligned} \det AB &= \det(E_1 E_2 \cdots E_k B) = (\det E_1)(\det E_2) \cdots (\det(E_k)(\det B) \\ &= (\det A)(\det B). \end{aligned}$$

Theorem

- If A is an invertible $n \times n$ matrix, then

$$\det A^{-1} = \frac{1}{\det A}.$$

- Reason: Since $AA^{-1} = I_n$, we have

$$1 = \det I_n = \det(AA^{-1}) = (\det A)(\det A^{-1}).$$

Definition

- If A is an $n \times n$ matrix, we call the matrix

$$\operatorname{adj} A = [C_{ij}]^T$$

  the *adjoint* of A.
- In other words, the (i, j)-entry of $\operatorname{adj} A$ is C_{ji}.

Example

- If
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$
then
$$\text{adj } A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$
- Note: $\det A = -2$ and
$$A(\text{adj } A) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Example

- If
$$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 2 & 3 \\ -4 & 1 & 5 \end{bmatrix},$$
then
$$\text{adj } A = \begin{bmatrix} 7 & -7 & 7 \\ -9 & 19 & -11 \\ 4 & -10 & 8 \end{bmatrix}^T = \begin{bmatrix} 7 & -9 & 4 \\ -7 & 19 & -10 \\ 7 & -11 & 8 \end{bmatrix}.$$
- Note: $\det A = 14$, and
$$A(\text{adj } A) = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix}.$$

Theorem

- If $A = [a_{ij}]$ is an $n \times n$ matrix and we let $A(\text{adj } A) = [b_{ij}]$, then
$$b_{ij} = a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = \begin{cases} \det A, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$
- Reason:
  - If $i = j$, b_{ij} is the cofactor expansion of $\det A$ along the ith row.
  - If $i \neq$j, b_{ij} is the cofactor expansion of the determinant of the matrix obtained from A by replacing the jth row by the ith row.
- Note: we may write this as $A(\text{adj } A) = (\det A)I_n$.
- We may also show that $(\text{adj } A)A = (\det A)I_n$.

Theorem

- If A is an $n \times n$ matrix with $\det A \neq 0$, then
$$A^{-1} = \frac{1}{\det A} \text{adj } A.$$

Example

- We saw previously that if

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 2 & 3 \\ -4 & 1 & 5 \end{bmatrix},$$

then

$$\text{adj}\, A = \begin{bmatrix} 7 & -9 & 4 \\ -7 & 19 & -10 \\ 7 & -11 & 8 \end{bmatrix}.$$

- Hence, since $\det A = 14$,

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 7 & -9 & 4 \\ -7 & 19 & -10 \\ 7 & -11 & 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{9}{14} & \frac{2}{7} \\ -\frac{1}{2} & \frac{19}{14} & -\frac{5}{7} \\ \frac{1}{2} & -\frac{11}{14} & \frac{4}{7} \end{bmatrix}.$$

Cramer's Rule

- Suppose $A = [a_{ij}]$ is an $n \times n$ matrix and $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$.
- If A is invertible, then the unique solution to $AX = B$ is

$$X = A^{-1}B = \frac{1}{\det A}(\text{adj}\, A)B = \frac{1}{\det A}\begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}.$$

Cramer's Rule (cont'd)

- Let $A_j(B)$ be the matrix obtained by replacing the jth column of A by B.
- Then we may rewrite the above as

$$\begin{aligned} x_1 &= \frac{\det A_1(B)}{\det A} \\ x_2 &= \frac{\det A_2(B)}{\det A} \\ \vdots \quad & \quad \vdots \\ x_n &= \frac{\det A_n(B)}{\det A}. \end{aligned}$$

Example

- If

$$\begin{aligned} 4x + 3y &= 6 \\ x - 5y &= 3, \end{aligned}$$

then

$$x = \frac{\det \begin{bmatrix} 6 & 3 \\ 3 & -5 \end{bmatrix}}{\det \begin{bmatrix} 4 & 3 \\ 1 & -5 \end{bmatrix}} = \frac{-30-9}{-20-3} = \frac{39}{23}$$

$$y = \frac{\det \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix}}{\det \begin{bmatrix} 4 & 3 \\ 1 & -5 \end{bmatrix}} = \frac{12-6}{-20-3} = -\frac{6}{23}.$$

Determinants in Octave

- If A is an $n \times n$ matrix, the command `det(A)` will compute $\det A$ in Octave.