Mathematics 160: Lecture 13

The Eigenvalue Problem

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Example

- Problem: rank web sites by importance.
- Suppose we want to rank n web sites by their importance. Let x_i be the importance of web site i.
- One method: the importance of a web site is proportional to the sum of the importances of the web sites which link to it.
- For example: suppose sites 3, 4, 21, and 52 link to site 8. Then, for some constant $\lambda > 0$, we want

$$x_3 + x_4 + x_{21} + x_{52} = \lambda x_8.$$

- Note: we want the same λ for all such equations.

Example (cont'd)

- Let $A = [a_{ij}]$ be an $n \times n$ matrix with

$$a_{ij} = \begin{cases} 1, & \text{if site } j \text{ links to site } i, \\ 0, & \text{if site } j \text{ does not link to site } i. \end{cases}$$

- If we let $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$, then we want to find λ and X such that

$$AX = \lambda X.$$

- Note: when $\lambda = 1$, X is a fixed vector.
- Note: equivalently, we want $(\lambda I - A)X = O$.

Definitions

- Suppose A is an $n \times n$ matrix. If λ is a scalar and X is a nonzero $n \times 1$ column matrix such that

$$AX = \lambda X,$$

then we say λ is an *eigenvalue* with *eigenvector* X.

- Note: the fixed vector for a Markov chain is an eigenvector with eigenvalue 1.

Characteristic polynomials

- $AX = \lambda X$ if and only if $(\lambda I - A)X = O$.
- $(\lambda I - A)X = O$ has a nontrivial solution if and only if $\lambda I - A$ is not invertible
- $\lambda I - A$ is invertible if and only if $\det(\lambda I - A) \neq 0$.
- Note: $\det(\lambda I - A)$ is a polynomial of degree n in the variable λ. We call

$$c_A(\lambda) = \det(\lambda I - A)$$

  the *characteristic polynomial* of A.
- Hence: the eigenvalues of an $n \times n$ matrix A are the roots of the characteristic polynomial $c_A(\lambda)$.
- Recall: a polynomial of degree n has n roots, some of which may be complex and some of which may be repeated.

Example

- Suppose

$$A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}.$$

- Then

$$\begin{aligned} c_A(\lambda) &= \det(\lambda I - A) \\ &= \det \begin{bmatrix} \lambda - 1 & -4 \\ -1 & \lambda + 2 \end{bmatrix} \\ &= (\lambda - 1)(\lambda + 2) - 4 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda + 3)(\lambda - 2). \end{aligned}$$

- Hence A has two eigenvalues: $\lambda_1 = -3$ and $\lambda_2 = 2$.

Example (cont'd)

- To find the associated eigenvectors, we need to find nontrivial solutions to the homogeneous equations $(-3I - A)X = O$ and $(2I - A)X = O$.
- For $\lambda_1 = -3$, we have

$$\begin{bmatrix} -4 & -4 & 0 \\ -1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- If we let $x_2 = t$, then $x_1 = -t$, so our solutions are

$$X = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

- In particular,

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

  is an eigenvector corresponding to the eigenvalue $\lambda_1 = -3$.

Example (cont'd)

- For $\lambda_2 = 2$, we have

$$\begin{bmatrix} 1 & -4 & 0 \\ -1 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- If we let $x_2 = t$, then $x_1 = 4t$, so our solutions are

$$X = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

- In particular,

$$X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

  is an eigenvector corresponding to the eigenvalue $\lambda_2 = 2$.

Example

- Suppose

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

- Then the characteristic polynomial is

$$\begin{aligned} c_A(\lambda) &= \det \begin{bmatrix} \lambda-2 & -2 & -1 \\ -1 & \lambda-3 & -1 \\ -1 & -2 & \lambda-2 \end{bmatrix} \\ &= (\lambda-2)((\lambda-3)(\lambda-2)-2)+2(-(\lambda-2)-1) \\ &\quad -(2+(\lambda-3)) \\ &= (\lambda-2)(\lambda^2-5\lambda+4)-3\lambda+3 \\ &= (\lambda-2)(\lambda-4)(\lambda-1)-3(\lambda-1) \\ &= (\lambda-1)(\lambda^2-6\lambda+8-3) \\ &= (\lambda-1)^2(\lambda-5). \end{aligned}$$

Example (cont'd)

- So the eigenvalues are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 5$.
- For $\lambda_1 = 1$, we have

$$\begin{bmatrix} -1 & -2 & -1 & 0 \\ -1 & -2 & -1 & 0 \\ -1 & -2 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- If we let $x_2 = t$ and $x_3 = s$, then $x_1 = -2t - s$.
- That is, our solutions are

$$X = t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Example (cont'd)

- In particular, with $t = 1$ and $s = 0$, we have the eigenvector

$$X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix},$$

and, with $t = 0$ and $s = 1$, we have the eigenvector

$$X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

both corresponding to the eigenvalue $\lambda_1 = \lambda_2 = 1$.

Example (cont'd)

- For the eigenvalue $\lambda_3 = 5$, we have

$$\begin{bmatrix} 3 & -2 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -2 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 3 & -2 & -1 & 0 \\ -1 & -2 & 3 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & -4 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- If we let $x_3 = t$, then $x_2 = t$ and $x_1 = t$.

Example (cont'd)

- That is, our solutions are

$$X = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

- In particular,

$$X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

  is an eigenvector corresponding to the eigenvalue $\lambda_3 = 5$.