Mathematics 160: Lecture 14

Diagonalization

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Example

- As in a previous example, let

$$A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}.$$

- We saw that

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue $\lambda_1 = -3$ and

$$X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue $\lambda_2 = 2$.

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Example (cont'd)

- Now let
$$P = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}.$$
- That is, P is the matrix whose columns are the eigenvectors of A.
- Then
$$\begin{aligned} P^{-1}AP &= -\frac{1}{5}\begin{bmatrix} 1 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \\ &= -\frac{1}{5}\begin{bmatrix} 1 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

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Definition

- We say an $n \times n$ matrix A is *diagonalizable* if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Diagonalization

- Suppose D is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $P^{-1}AP = D$.
- Then $AP = PD$.
- If $X_1, X_2, \ldots, X_n$ are the columns of P, then we have

$$A\begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

- Hence $AX_1 = \lambda_1 X_1$, $AX_2 = \lambda_2 X_2$, $\ldots$, $AX_n = \lambda_n X_n$.

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- Hence $AX_1 = \lambda_1 X_1$, $AX_2 = \lambda_2 X_2$, $\ldots$, $AX_n = \lambda_n X_n$.

Theorem

- An $n \times n$ matrix A is diagonalizable if and only if A has n eigenvectors $X_1, X_2, \ldots, X_n$ with the property that the matrix P whose columns are $X_1, X_2, \ldots, X_n$ is invertible.
- If A is diagonalizable, then, with P as above,

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where λ_i is the eigenvalue corresponding to the eigenvector X_i.

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Definitions

- We say an eigenvalue λ of an $n \times n$ matrix A has *multiplicity* m if $x - \lambda$ occurs exactly m times in the factorization of $c_A(x)$.
- If λ is an eigenvalue of an $n \times n$ matrix A, we call a set of basic solutions of the equation $(\lambda I_n - A)X = O$ *basic eigenvectors* corresponding to λ.
- Note: an eigenvalue of multiplicity m will have at most m basic eigenvectors.

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- In particular, if an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

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Example

- Suppose

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

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- From our previous work,

$$X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

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Example (cont'd)

- And

$$X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is a basic eigenvector for the eigenvalue $\lambda_3 = 5$.

- Then

$$P = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}.$$

- You may check that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

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Example

- Suppose

$$A = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 2 & 3 \\ 2 & -3 & -4 \end{bmatrix}.$$

- The characteristic polynomial is

$$\begin{aligned} c_A(\lambda) &= \det \begin{bmatrix} \lambda - 1 & 2 & 2 \\ 2 & \lambda - 2 & -3 \\ -2 & 3 & \lambda + 4 \end{bmatrix} \\ &= (\lambda - 1)((\lambda - 2)(\lambda + 4) + 9) - 2(2(\lambda + 4) - 6) \\ &\quad + 2(6 + 2(\lambda - 2)) \\ &= (\lambda - 1)(\lambda^2 + 2\lambda + 1) - (4\lambda + 4) + (4\lambda + 4) \\ &= (\lambda - 1)(\lambda + 1)^2. \end{aligned}$$

- So the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$.

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Example (cont'd)

- For $\lambda_1 = 1$, we have

$$\begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & -1 & -3 & 0 \\ -2 & 3 & 5 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ -2 & 3 & 5 & 0 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 2 & -1 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- So if we let $x_3 = t$, then $x_2 = -t$ and $x_1 = t$.

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- That is,

$$X = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

- In particular,

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- For the eigenvalue $\lambda_2 = \lambda_3 = -1$, we have

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Eigenvalues in Octave

- Given an $n \times n$ matrix A, the Octave command `eig(A)` returns the eigenvalues of A.
- The command `[P, D] = eig(A)` returns P as the matrix of eigenvectors and D as the corresponding diagonal matrix of eigenvalues.
- `P(:,j)` will return the *jth* column of P, that is, the jth eigenvector of A.
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