Mathematics 160: Lecture 15

Similar Matrices and Complex Numbers

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Powers

- Note: if D is the diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

then

$$D^m = \begin{bmatrix} \lambda_1^m & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^m & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n^m \end{bmatrix}.$$

Powers (cont'd)

- Suppose A is an $n \times n$ diagonalizable matrix. That is, there exists an invertible matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix.
- Then $A = PDP^{-1}$, and so

$$A^m = \underbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}_{m \text{ times}} = PD^mP^{-1}.$$

Example

- In a previous example, we saw that if

$$A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$$

and

$$P = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix},$$

then

$$P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Example (cont'd)

- Hence, for example,

$$A^9 = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -19,683 & 0 \\ 0 & 512 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3,936.6 & -15,746.4 \\ 102.4 & 102.4 \end{bmatrix}$$
$$= \begin{bmatrix} -3,527 & 16,156 \\ 4,039 & -15,644 \end{bmatrix}.$$

Definition

- We say $n \times n$ matrices A and B are *similar* if $B = P^{-1}AP$ for some invertible matrix P.
- Notation: we write $A \sim B$ if A and B are similar.
- Note: If $A \sim B$, then
  - $\det A = \det B$.
  - $c_A(x) = c_B(x)$.
  - A and B have the same eigenvalues.
- Reasons:
  - If $B = P^{-1}AP$, then

$$\det B = (\det P^{-1})(\det A)(\det P) = \det A.$$

  - If $A \sim B$, then $(\lambda I - B) \sim (\lambda I - A)$ since

$$P^{-1}(\lambda I - A)P = \lambda P^{-1}IP - P^{-1}AP = \lambda I - B.$$

Example

- Suppose the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

  is such that

$$a_{ij} = \begin{cases} 1, & \text{if web site } j \text{ links to web site } i, \\ 0, & \text{otherwise.} \end{cases}$$

Example (cont'd)

- Using Octave, we find one eigenvector with all positive entries:

$$X = \begin{bmatrix} 0.32569 \\ 0.25836 \\ 0.20181 \\ 0.30193 \\ 0.43490 \\ 0.40579 \\ 0.32524 \\ 0.23199 \\ 0.32010 \\ 0.28181 \end{bmatrix}.$$

- With this ranking, web site 5 is the most important site, site 6 is the second most important, site 1 the third most important, and so on.

Complex numbers

- Recall: we let i denote the number with the property that $i^2 = -1$.
- More generally, a number of the form $a + bi$, where a and b are real numbers, is a *complex number*.
- We call a the *real part* of $a + bi$ and b the *imaginary part* of $a + bi$.
- Geometrically, we may think of the complex number $a + bi$ as the point in the plane with coordinates (a, b).

Addition and multiplication of complex numbers

- If $z = a + bi$ and $w = c + di$, then
$$z + w = (a + c) + (b + d)i.$$
- Example: $(3 - 4i) + (6 + 2i) = 9 - 2i$.
- If $z = a + bi$ and $w = c + di$, then
$$zw = (a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$
- Example: $(3 + 4i)(2 - i) = (6 + 4) + (8 - 3)i = 10 + 5i$.

Complex conjugates

- If $z = a + bi$, we call $\bar{z} = a - bi$ the *conjugate* of z.
- Note: geometrically, $\bar{z}$ is obtained by reflecting z over the real axis.
- Example: if $z = 1 - 2i$, then $\bar{z} = 1 + 2i$.
- Properties: if z and w are complex numbers, then
  - $\overline{z + w} = \bar{z} + \bar{w}$.
  - $\overline{zw} = \bar{z}\bar{w}$.
  - z is real if and only if $\bar{z} = z$.
- Note: if $z = a + bi$, then
$$z\bar{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2.$$
- Hence: if we let $|z| = \sqrt{a^2 + b^2}$, then $|z|^2 = z\bar{z}$.
- We call $|z|$ the *modulus* of z. Geometrically, $|z|$ is the distance from z to the origin.

Inverses

- If $z = a + bi$, $z \neq 0$, then
$$z^{-1} = \frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$
- Example: if $z = 3 + 2i$, then
$$\frac{1}{z} = \frac{1}{3 + 2i} = \frac{1}{3 + 2i} \times \frac{3 - 2i}{3 - 2i} = \frac{3 - 2i}{9 + 4} = \frac{3}{13} - \frac{2}{13} i.$$
- Example:
$$\frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} = \frac{-i}{1} = -i.$$

Division

- We can use the same idea for division.
- For example,

$$\frac{1+2i}{3-5i} = \frac{1+2i}{3-5i} \times \frac{3+5i}{3+5i} = \frac{-7+11i}{9+25} = -\frac{7}{34} + \frac{11}{34}i.$$