Mathematics 160: Lecture 16

Complex Eigenvalues

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Conjugate pairs of eigenvalues

- Suppose $\lambda = a + bi$ is complex eigenvalue of the real $n \times n$ matrix A.
- Let $X = X_1 + iX_2$ be an eigenvector for λ, where X_1 an X_2 are both real.
- Now
$$AX = A(X_1 + iX_2) = AX_1 + iAX_2$$
and
$$AX = \lambda X = (a + bi)(X_1 + iX_2) = (aX_1 - bX_2) + i(bX_1 + aX_2).$$
- Equating the real and imaginary parts, we have
$$AX_1 = aX_1 - bX_2 \text{ and } AX_2 = bX_1 + aX_2.$$

Conjugate pairs (cont'd)

- It follows that
$$\begin{aligned} A\bar{X} &= A(X_1 - iX_2) \\ &= AX_1 - iAX_2 \\ &= (aX_1 - bX_2) - i(bX_1 + aX_2) \\ &= (a - bi)(X_1 - iX_2) \\ &= \bar{\lambda}\bar{X}. \end{aligned}$$
- That is: if λ is an eigenvalue of $n \times n$ real matrix A with eigenvector X, then $\bar{\lambda}$ is a an eigenvalue of A with eigenvector $\bar{X}$.

Example

- Suppose
$$A = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}.$$
- The characteristic polynomial of A is
$$\begin{aligned} c_A(\lambda) &= \det \begin{bmatrix} \lambda - 6 & 1 \\ -5 & \lambda - 2 \end{bmatrix} \\ &= (\lambda - 6)(\lambda - 2) + 5 \\ &= \lambda^2 - 8\lambda + 17. \end{aligned}$$
- So the eigenvalues of A are
$$\lambda_1 = \frac{8 - \sqrt{64 - 68}}{2} = 4 - i \text{ and } \lambda_2 = 4 + i.$$

Example (cont'd)

- To find an eigenvector corresponding to $\lambda_2 = 4 + i$, we have

$$\begin{bmatrix} -2+i & 1 & 0 \\ -5 & 2+i & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & -2-i & 0 \\ -5 & 2+i & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & -2-i & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Hence if we let $x_2 = t$, we have $x_1 = \left(\frac{2}{5} + \frac{1}{5}i\right) t$.

Example (cont'd)

- That is,

$$X = t \begin{bmatrix} \frac{2}{5} + \frac{1}{5}i \\ 1 \end{bmatrix}.$$

- Hence, taking $t = 5$,

$$X_2 = \begin{bmatrix} 2+i \\ 5 \end{bmatrix}$$

is an eigenvector corresponding to $\lambda_2 = 4 + i$.

- It follows that

$$X_1 = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$$

is an eigenvector corresponding to $\lambda_1 = 4 - i$.

Example

- Suppose

$$A = \begin{bmatrix} 2 & 5 & 1 \\ -5 & -6 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

- The characteristic polynomial is

$$\begin{aligned} c_A(\lambda) &= \det \begin{bmatrix} \lambda-2 & -5 & -1 \\ 5 & \lambda+6 & -4 \\ 0 & 0 & \lambda-2 \end{bmatrix} \\ &= (\lambda-2)((\lambda-2)(\lambda+6)+25) \\ &= (\lambda-2)(\lambda^2+4\lambda+13). \end{aligned}$$

- Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = -2 - 3i$, and $\lambda_3 = -2 + 3i$.

Example (cont'd)

- For $\lambda_1 = 2$, we have

$$\begin{bmatrix} 0 & -5 & -1 & 0 \\ 5 & 8 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{8}{5} & -\frac{4}{5} & 0 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- If we let $x_3 = t$, then $x_2 = -\frac{1}{5}t$ and $x_1 = \frac{28}{25}t$.
- If we take $t = 25$, we have

$$X_1 = \begin{bmatrix} 28 \\ -5 \\ 25 \end{bmatrix}$$

as an eigenvector corresponding to $\lambda_1 = 2$.

Example (cont'd)

- For $\lambda_2 = -2 - 3i$, we have

$$\begin{bmatrix} -4-3i & -5 & -1 & 0 \\ 5 & 4-3i & -4 & 0 \\ 0 & 0 & -4-3i & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 25 & 20-15i & 4-3i & 0 \\ 5 & 4-3i & -4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 5 & 4-3i & -4 & 0 \\ 0 & 0 & 24+3i & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & 4-3i & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- If we let $x_2 = t$, then $x_3 = 0$ and $x_1 = \left(-\frac{4}{5} + \frac{3}{5}i\right) t$.

Example (cont'd)

- With $t = 5$, we have

$$X_2 = \begin{bmatrix} -4+3i \\ 5 \\ 0 \end{bmatrix}$$

as an eigenvector corresponding to $\lambda_2 = -2 - 3i$.
- It follows that

$$X_3 = \begin{bmatrix} -4-3i \\ 5 \\ 0 \end{bmatrix}$$

is an eigenvector corresponding to $\lambda_3 = -2 + 3i$.

Symmetric matrices

- Recall: we say an $n \times n$ matrix is symmetric if $A^T = A$.
- Suppose A is a real symmetric matrix and let λ be an eigenvalue of A.
- Let

$$X = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

be an eigenvector corresponding to λ.
- Let

$$c = X^T \bar{X} = z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2.$$

- Note: c is real and $c > 0$.

Symmetric matrices (cont'd)

- Now

$$\lambda c = \lambda\left(X^T \bar{X}\right) = (\lambda X)^T \bar{X} = (AX)^T \bar{X} = X^T A \bar{X}$$
$$= X^T(\bar{\lambda}\bar{X}) = \bar{\lambda}\left(X^T \bar{X}\right) = \bar{\lambda} c.$$

- Hence $\lambda = \bar{\lambda}$
- That is, λ is real.
- Conclusion: The eigenvalues of a symmetric real matrix are real.
- We shall see later that in fact every symmetric matrix is diagonalizable (part of the *Principal Axis Theorem*).