Mathematics 160: Lecture 17

Linear Recurrences

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Definition

- We say a sequence x_0, x_1, x_2, ... is generated by a kth-order *recursion relation* if
$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}).$$
for $n = k, k+1, k+2, \ldots$.
- We say the recurrence is *linear* if for some constants a_0, a_1, ... a_{k-1},
$$f(x_n, x_{n+1}, \ldots, x_{n+k-1}) = a_0 x_n + a_1 x_{n+1} + \cdots + a_{k-1} x_{n+k-1}.$$

Example

- Given a constant a, $x_{n+1} = ax_n$ is a first-order linear recurrence relation.
- We have
$$\begin{aligned} x_1 &= ax_0, \\ x_2 &= ax_1 = a^2 x_0, \\ x_3 &= ax_2 = a^3 x_0, \\ x_4 &= ax_3 = a^4 x_0, \end{aligned}$$
and so on.
- In general, $x_n = a^n x_0$.
- In particular, we have *exponential growth* if $a > 1$ and *exponential decay* if $0 < a < 1$.

Example

- Starting with $x_0 = 1$ and $x_1 = 1$, the second-order linear recurrence relation
$$x_{n+2} = x_n + x_{n+1}$$
generates the *Fibonacci sequence*: $1, 1, 2, 3, 5, 8, 13, 21, \ldots$.
- Let
$$V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and, for } n = 1, 2, 3, \ldots, \; V_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}.$$
- Then $V_{n+1} = AV_n$ where
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$
- Iterating, we have $V_n = A^n V_0$.

Second-order recurrence relations

- Consider a sequence $x_0, x_1, x_3, \ldots$ generated by the second-order linear recurrence relation

$$x_{n+2} = ax_n + bx_{n+1}.$$

- Let, for $n = 0, 1, 2, \ldots$,

$$V_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}.$$

- Then

$$V_{n+1} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ ax_n + bx_{n+1} \end{bmatrix} = AV_n.$$

- Iterating, we have $V_n = A^n V_0$.

Second-order recurrence relations (cont'd)

- Now suppose A is diagonalizable.
- That is, suppose there exists a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.
- Then $A^n = PD^nP^{-1}$.
- Now if λ_1 and λ_2 are the eigenvalues of A, then

$$D^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}.$$

- It follows that the entries of $V_n = A^n V_0$ are linear combinations of λ_1^n and λ_2^n.
- In particular, for some constants c_1 and c_2,

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n.$$

Example

- For the Fibonacci sequence $x_{n+2} = x_n + x_{n+1}$,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

- The characteristic polynomial of A is

$$c_A(\lambda) = \det \begin{bmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{bmatrix} = \lambda^2 - \lambda - 1.$$

- So the eigenvalues are

$$\lambda_1 = \frac{1-\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{1+\sqrt{5}}{2}.$$

- Hence, for some constants c_1 and c_2,

$$x_n = c_1\left(\frac{1-\sqrt{5}}{2}\right)^n + c_2\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

Example (cont'd)

- Evaluating for $x_0 = 1$ and $x_1 = 1$, we have

$$\begin{aligned} c_1 + c_2 &= 1 \\ \lambda_1 c_1 + \lambda_2 c_2 &= 1. \end{aligned}$$

- Using Cramer's rule, we have

$$\begin{aligned} c_1 &= \frac{\lambda_2 - 1}{\lambda_2 - \lambda_1} = -\frac{\lambda_1}{\sqrt{5}} \\ c_2 &= \frac{1 - \lambda_1}{\lambda_2 - \lambda_1} = \frac{\lambda_2}{\sqrt{5}}. \end{aligned}$$

- Hence

$$x_n = \frac{1}{\sqrt{5}}(\lambda_2^{n+1} - \lambda_1^{n+1}).$$

Example (cont'd)

- For example,

$$x_{20} = \frac{\lambda_2^{21} - \lambda_1^{21}}{\sqrt{5}} = \frac{(5473\sqrt{5} + 12238) - (-5473\sqrt{5} + 12238)}{\sqrt{5}} = 10946.$$

- Note: by themselves,

$$\frac{\lambda_2^{21}}{\sqrt{5}} = 10945.9999817284$$

and

$$\frac{\lambda_1^{21}}{\sqrt{5}} = -0.0000182715.$$

Example (cont'd)

- In general, $\frac{\lambda_2^{n+1}}{\sqrt{5}}$ provides a good approximation for x_n when n is large.
- This follows from
  - noting that $|\lambda_1| < |\lambda_2|$, and
  - $x_n = \frac{\lambda_2^{n+1}}{\sqrt{5}}\left(1 - \left(\frac{\lambda_1}{\lambda_2}\right)^{n+1}\right)$.
- We call λ_2 the *dominant* eigenvalue.

Theorem

- Suppose

$$x_{n+k} = a_0 x_n + a_1 x_{n+1} + \cdots + a_{k-1} x_{n+k-1}.$$

- Let

$$V_n = \begin{bmatrix} x_n \\ x_{n+1} \\ \vdots \\ x_{n+k-1} \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{k-1} \end{bmatrix}$$

so that $V_{n+1} = AV_n$.

- If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct real eigenvalues of A, then

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_k\lambda_k^n$$

for some constants $c_1, c_2, \ldots, c_k$.