Mathematics 160: Lecture 18

Vectors

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October 10, 2011

Two- and three-dimensional space

- Recall: a point P in the plane may be specified by an ordered pair (a, b) of real numbers.
- Similarly, a point P in three-space may be specified by an ordered triple (a, b, c) of real numbers.
- That is, letting $\mathbb{R}$ denote the set of real numbers, we may identify the plane with
$$\mathbb{R}^2 = \{(a, b) : a, b \in \mathbb{R}\}$$
and three-space with
$$\mathbb{R}^3 = \{(a, b, c) : a, b, c \in \mathbb{R}\}.$$

Geometric vectors

- Given points P and Q in $\mathbb{R}^2$ or $\mathbb{R}^3$, we let $\overrightarrow{PQ}$ denote the directed line segment from P to Q, which we call a *geometric vector*.
- If O is the origin and $P = (a, b, c)$, then we may identify the geometric vector $\overrightarrow{OP}$ with the point (a, b, c).
- If $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$, then $\overrightarrow{PQ}$ is the same as the vector $\overrightarrow{OR}$ where $R = (a_2 - a_1, b_2 - b_1, c_2 - c_1)$. We call the latter the *standard form* of $\overrightarrow{PQ}$.

Vectors and column matrices

- We may identify a vector in standard form with a column matrix.
- That is, if $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$, we may write
$$\overrightarrow{PQ} = \begin{bmatrix} a_2 - a_1 \\ b_2 - b_1 \\ c_2 - c_1 \end{bmatrix}.$$

Algebraic operations

- If α is a scalar,

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},$$

then

$$\alpha\vec{u} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \end{bmatrix},$$

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix},$$

$$\vec{u} - \vec{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}.$$

Geometric interpretations

- Geometrically, $\vec{u} + \vec{v}$ is the diagonal of the parallelogram with adjacent sides $\vec{u}$ and $\vec{v}$.
- Geometrically, $\vec{u} - \vec{v}$ is the directed line segment from the tip of $\vec{v}$ to the tip of $\vec{u}$.
- Geometrically, $\alpha\vec{u}$ is $\vec{u}$ stretched by a factor of $|\alpha|$, in the same direction as $\vec{u}$ if $\alpha > 0$ and in the opposite direction if $\alpha < 0$.

Length

- Notation: we let $\|\vec{v}\|$ denote the length of a vector, which we also call the *norm* of $\vec{v}$.
- If

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix},$$

then, by the Pythagorean theorem,

$$\|\vec{v}\| = \sqrt{x^2 + y^2}.$$

Length (cont'd)

- If

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

let

$$\vec{u} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

- Then, again by the Pythagorean theorem,

$$\|\vec{v}\| = \sqrt{\|\vec{u}\|^2 + \|\vec{v} - \vec{u}\|^2} = \sqrt{x^2 + y^2 + z^2}.$$

Example

- If $P = (-1, 2, 3)$ and $Q = (1, 3, 1)$, then

$$\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

and $\|\overrightarrow{PQ}\| = \sqrt{4 + 1 + 4} = 3$.

Properties of the norm

- Notation: let $\vec{0}$ be the vector with all coordinates equal to 0.
- Then
  - $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$.
  - For any scalar α, $\|\alpha\vec{v}\| = |\alpha|\|\vec{v}\|$.
  - If $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, then

$$\|\overrightarrow{PQ}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Parallel vectors

- Intuitively, vectors $\vec{v}$ and $\vec{w}$ are parallel if they have the same or opposite direction.
- Definition: we say nonzero vectors $\vec{v}$ and $\vec{w}$ are *parallel* if one is a scalar multiple of the other.