Mathematics 160: Lecture 19

Dot Products

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Higher dimensions

- We let $\mathbb{R}^n$ denote the space of all ordered n-tuples of real numbers. That is,
$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}.$$
- We may identify a point $P = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ with the vector
$$\overrightarrow{OP} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$
- Addition, subtaction, and scalar multiplication is the same as for matrices (or for vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$).
- If $\vec{v} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$, then the *length* or *norm* of $\vec{v}$ is
$$\|\vec{v}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Example

- If
$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} -2 \\ 1 \\ 3 \\ -1 \end{bmatrix},$$
then
$$4\vec{v} = \begin{bmatrix} 4 \\ 8 \\ 12 \\ 16 \end{bmatrix}, 2\vec{v} - 3\vec{w} = \begin{bmatrix} 8 \\ 1 \\ -3 \\ 11 \end{bmatrix},$$
and $\|\vec{v}\| = \sqrt{1 + 4 + 9 + 16} = \sqrt{30}$.

Definition

- The *dot product* of vectors
$$\vec{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$
in $\mathbb{R}^n$ is the scalar
$$\vec{u} \cdot \vec{v} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$
- Example: if
$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix},$$
then
$$\vec{u} \cdot \vec{v} = -2 + 6 - 12 = -8.$$

Properties

- Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$ and a is a scalar. Then
  - $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$,
  - $\vec{u} \cdot \vec{0} = 0$,
  - $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$,
  - $(a\vec{u}) \cdot \vec{v} = a(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (a\vec{v})$,
  - $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$,
  - $\vec{u} \cdot (\vec{v} - \vec{w}) = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w}$.

Example

- From the properties, it follows that for any vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$,

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})\\ &= \vec{u}\cdot\vec{u} + 2(\vec{u}\cdot\vec{v}) + \vec{v}\cdot\vec{v}\\ &= \|\vec{u}\|^2 + 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2.\end{aligned}$$

Angles

- Similarly, $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 - 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2$.
- Note: $\vec{u}$, $\vec{v}$, and $\vec{u}-\vec{v}$ are the sides of a triangle.
- Now suppose $\vec{u}$ and $\vec{v}$ are nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and θ is the angle between $\vec{u}$ and $\vec{v}$.
- By the law of cosines,

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos(\theta).$$

- Hence we must have $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos(\theta)$, or

$$\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}.$$

Example

- If θ is the smallest angle, measured in the counterclockwise direction, between

$$\vec{u} = \begin{bmatrix}1\\2\\3\end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix}2\\-1\\4\end{bmatrix},$$

  then

$$\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{2-2+12}{\sqrt{14}\sqrt{21}} = \frac{12}{7\sqrt{6}}.$$

- Hence

$$\theta = \cos^{-1}\left(\frac{12}{7\sqrt{6}}\right) \approx 0.7649.$$

Cauchy-Schwarz Inequality

- Given vectors $\vec{v}$ and $\vec{u}$ in $\mathbb{R}^n$, define

$$p(t) = \|t\vec{u} - \vec{v}\|^2 = t^2\|\vec{u}\|^2 - 2t(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2.$$

- Since $p(t) \geq 0$ for all t, it follows, from the quadratic formula, that

$$4(\vec{u} \cdot \vec{v})^2 - 4\|\vec{u}\|^2\|\vec{v}\|^2 \leq 0.$$

- Hence

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|,$$

  which we call the *Cauchy-Schwarz Inequality*.

Cauchy-Schwarz Inequality (cont'd)

- Note: from the proof, we see that

$$|\vec{u} \cdot \vec{v}| = \|\vec{u}\|\|\vec{v}\|$$

  if and only if $\|t\vec{u} - \vec{v}\| = 0$ for some value of t.
- Hence we have equality in the Cauchy-Schwarz inequality if and only if $\vec{u}$ and $\vec{v}$ are parallel.

Definition

- Given vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, we call

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$$

  the *angle* between $\vec{u}$ and $\vec{v}$.
- Note: by the definition of the inverse cosine function, this means that θ is the angle such that $0 \leq \theta \leq \pi$ and

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{u}\|}.$$

Orthogonality

- Note: if the angle θ between the nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ or $\mathbb{R}^2$ is $\frac{\pi}{2}$, then $\cos(\theta) = 0$, and so $\vec{u} \cdot \vec{v} = 0$.
- More generally, we say vectors in $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are *orthogonal*, or *perpendicular*, if $\vec{u} \cdot \vec{v} = 0$.
- Note: the definition implies that $\vec{0}$ is orthogonal to any vector $\vec{u}$ in $\mathbb{R}^n$.