Mathematics 160: Lecture 2

The Algebra of Matrices

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Algebraic operations

- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices and c is a *scalar* (that is, a real number), then we define
  - $A + B = [a_{ij} + b_{ij}]$,
  - $cA = [ca_{ij}]$.
- That is, we add two matrices of the same size by adding their respective entries, term by term, and we multiply a matrix by a scalar by multiplying each entry by the scalar.

Examples

- If

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 10 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 5 \\ -8 & 13 \\ 1 & 2 \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} 1 & 7 \\ -5 & 9 \\ 11 & -1 \end{bmatrix} \text{ and } 3A = \begin{bmatrix} 3 & 6 \\ 9 & -12 \\ 30 & -9 \end{bmatrix}.$$

Transposition

- If $A = [a_{ij}]$ is an $m \times n$ matrix, then we call the $n \times m$ matrix

$$A^T = [b_{ij}] \text{ where } b_{ij} = a_{ji}$$

the *transpose* of A.
- That is, we obtain A^T by reflecting A across its *main diagonal* (the entries a_{11}, a_{22}, ..., a_{mm}).
- That is, the columns of A are the rows of A^T and the rows of A are the columns of A^T.

Examples

- If
$$A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 5 & 4 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 0 & 4 \end{bmatrix}$$
- If
$$B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \text{ then } B^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.$$
- B is an example of a *column matrix* and B^T is an example of a *row matrix*.

Properties of transposes

- If A and B are an $m \times n$ matrices and c is a scalar, then
  - $(A^T)^T = A$,
  - $(cA)^T = cA^T$,
  - $(A+B)^T = A^T + B^T$.

Symmetric matrices

- We say a matrix A is *symmetric* if $A^T = A$.
- A symmetric matrix is necessarily a *square matrix*, that is, a matrix with the same number of rows and columns.
- Examples: If
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$
then A is symmetric and B is not symmetric.

Some Octave

- In Octave, the commands
```
A = [1 2 3; -1 5 7]
```
and
```
B = [-1 3 8; -10 9 1]
```
create the 2×3 matrices
$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 5 & 7 \end{bmatrix}$$
and
$$B = \begin{bmatrix} -1 & 3 & 8 \\ -10 & 9 & 1 \end{bmatrix}$$

Some Octave (cont'd)

- Then the commands `A + B`, `4*B`, and `4*A - 7*B` do the obvious things.
- The command `A'` will find A^T.
- The command `A = diag([1 2 3])` will create the *diagonal* matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Systems of linear equations

- We call an equation of the form

  $$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

  a *linear equation*.
- Note: Frequently, we will write the variables in such an equation as a column matrix. That is, we may denote the variables as

  $$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T.$$

- We may then denote a solution $x_1 = s_1$, $x_2 = s_2$, ..., $x_n = s_n$ by $X = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix}^T$.

Example

- If $2x + y = 5$, then $y = 5 - 2x$.
- It follows that for any real number s, $x = s$ and $y = 5 - 2s$ satisfies the equation.
- That is, for any value of s,

  $$X = \begin{bmatrix} s \\ 5 - 2s \end{bmatrix} = s\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

  is a solution of $2x + y = 5$.
- We call s a *parameter* for the set of solutions to the equation.
- Note: the set of solutions is infinite. Geometrically, the set of solutions is a line in the plane.

Systems

- We often want to find simultaneous solutions to two or more linear equations.
- Example: To solve the system

  $$\begin{aligned} 2x - y &= 5 \\ x + 3y &= 2, \end{aligned}$$

  we note that it is equivalent to the the system (obtained by multiplying the second equation by -2 and adding it to the first)

  $$\begin{aligned} -7y &= 1 \\ x + 3y &= 2. \end{aligned}$$

- This gives us the solution $y = -\frac{1}{7}$ and $x = \frac{17}{7}$, that is,

  $$X = \begin{bmatrix} \frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}.$$

Example

- The system

$$\begin{aligned} 4x + y - z &= 6 \\ x - 3y + z &= 5 \end{aligned}$$

is equivalent to the system

$$\begin{aligned} 13y - 5z &= -14 \\ x - 3y + z &= 5. \end{aligned}$$

- If we let $z = s$, where s can be any real number, then

$$y = \frac{5}{13}s - \frac{14}{13} \text{ and } x = \frac{2}{13}s + \frac{23}{13}.$$

Example (cont'd)

- Hence the solutions are

$$X = \begin{bmatrix} \frac{2}{13}s + \frac{23}{13} \\ \frac{5}{13}s - \frac{14}{13} \\ s \end{bmatrix} = s \begin{bmatrix} \frac{2}{13} \\ \frac{5}{13} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{23}{13} \\ -\frac{14}{13} \\ 0 \end{bmatrix}.$$

- Note: the solution set is infinite. Geometrically, this is a line in three-space.

Example

- The system of equations

$$\begin{aligned} 3x - 4y &= 12 \\ -6x + 8y &= 13 \end{aligned}$$

is *inconsistent* because there are no solutions: if $3x - 4y = 12$, then, multiplying by -2, we have to have $-6x + 8y = -24$.
- We can also see this by multiplying the first equation by 2 and adding it to the second to obtain the equivalent system

$$\begin{aligned} 3x - 4y &= 12 \\ 0 &= 37, \end{aligned}$$

which clearly has no solutions.

Number of solutions

- Note: we have seen an example of a system of linear equations having exactly one solution, an example with an infinite number of solutions, and an example with no solutions. We shall see that these are the only possibilities for the number of solutions to a system of linear equations.