Mathematics 160: Lecture 20

Projections

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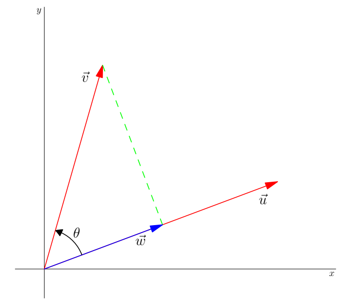
Projections in $\mathbb{R}^2$

- Consider two non-zero vectors, $\vec{u}$ and $\vec{v}$, in $\mathbb{R}^2$, and let θ be the angle between them.



- If $\vec{w}$ is the orthogonal projection of $\vec{v}$ onto $\vec{u}$, either in the same direction as $\vec{u}$ if $0 \leq \theta \leq \frac{\pi}{2}$, or in the opposite direction if $\frac{\pi}{2} < \theta \leq \pi$, then
$$\|\vec{w}\| = \|\vec{v}\| |\cos(\theta)| = \frac{|\vec{v} \cdot \vec{u}|}{\|\vec{u}\|}.$$

Projections in $\mathbb{R}^2$ (cont'd)

- Note: $\vec{e} = \frac{1}{\|\vec{u}\|}\vec{u}$ is a vector of unit length in the same direction as $\vec{u}$.
- We call the vector
$$\frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|}\vec{e} = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u}$$
the *projection of* $\vec{v}$ *on* $\vec{u}$, which we denote $\operatorname{proj}_{\vec{u}} \vec{v}$.
- We call
$$\frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|}$$
the *component of* $\vec{v}$ *in the direction of* $\vec{u}$.

Example

- Let
$$\vec{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$
- Then
$$\operatorname{proj}_{\vec{u}} \vec{v} = \frac{7}{10}\vec{u} = \begin{bmatrix} \frac{21}{10} \\ \frac{7}{10} \end{bmatrix}.$$
- The component of $\vec{v}$ in the direction of $\vec{u}$ is $\frac{7}{\sqrt{10}}$

Projections in $\mathbb{R}^n$

- Now suppose $\vec{v}$ and $\vec{u}$ are nonzero vectors in $\mathbb{R}^n$.
- We wish to write $\vec{v} = \vec{v}_1 + \vec{v}_2$, where $\vec{v}_1$ is parallel to $\vec{u}$ and $\vec{v}_2$ is orthogonal to $\vec{u}$.
- That is, we want to find a scalar t such that $\vec{v}_1 = t\vec{u}$ and $\vec{v}_2 = \vec{v} - \vec{v}_1$ is orthogonal to $\vec{u}$.
- So we want
$$0 = (\vec{v} - t\vec{u}) \cdot \vec{u} = \vec{v} \cdot \vec{u} - t\vec{u} \cdot \vec{u} = \vec{v} \cdot \vec{u} - t\|\vec{u}\|^2.$$
- Hence we must have
$$t = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}.$$

Definition

- Suppose $\vec{v}$ and $\vec{u}$ are vectors in $\mathbb{R}^n$ with $\vec{u} \neq \vec{0}$.
- We call
$$\operatorname{proj}_{\vec{u}} \vec{v} = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u}$$
the *projection of $\vec{v}$ on $\vec{u}$*, and we call
$$\frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|}$$
the *component of $\vec{v}$ in the direction of $\vec{u}$*.
- Note: if $\|\vec{u}\| = 1$, that is, $\vec{u}$ is a *unit vector*, then
$$\operatorname{proj}_{\vec{u}} \vec{v} = (\vec{v} \cdot \vec{u})\vec{u}$$
and the component of $\vec{v}$ in the direction of $\vec{u}$ is $\vec{v} \cdot \vec{u}$.

Example

- If
$$\vec{v} = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \vec{u} = \begin{bmatrix} -3 \\ 1 \\ -2 \\ -4 \end{bmatrix},$$
then
$$\operatorname{proj}_{\vec{u}} \vec{v} = -\frac{15}{30}\vec{u} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \\ 2 \end{bmatrix}$$
and the component of $\vec{v}$ in the direction of $\vec{u}$ is $-\frac{1}{2}\sqrt{30}$.

Example

- Suppose $\vec{v}$ is a vector in $\mathbb{R}^n$ and, for $j = 1, 2, \ldots, n$, $\vec{e}_j$ is a vector in $\mathbb{R}^n$ with
$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th row}$$
- Then
$$\operatorname{proj}_{\vec{e}_j} \vec{v} = (\vec{v} \cdot \vec{e}_j)\vec{e}_j = x_j\vec{e}_j,$$
and the component of $\vec{v}$ in the direction of $\vec{e}_j$ is just the jth coordinate of $\vec{v}$, namely, x_j.