Mathematics 160: Lecture 21

Lines

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Lines in $\mathbb{R}^2$.

- Suppose ℓ is a line in $\mathbb{R}^2$ passing through the points $\vec{p}_0$ and $\vec{q}_0$.
- Note: $\vec{d} = \vec{q}_0 - \vec{p}_0$ is a vector parallel to ℓ.
- Note: $\vec{p}$ is on ℓ if and only if $\vec{p} - \vec{p}_0$ is parallel to $\vec{d}$.
- That is, $\vec{p}$ is on ℓ if and only if $\vec{p} - \vec{p}_0 = t\vec{d}$ for some scalar t.
- That is, $\vec{p}$ is on ℓ if and only if $\vec{p} = \vec{p}_0 + t\vec{d}$ for some scalar t.

Definition

- Given $\vec{p}_0$ and $\vec{d}$ in $\mathbb{R}^n$, we call the set of all points $\vec{p}_0 + t\vec{d}$, where t is a scalar, the *line* through $\vec{p}_0$ with direction $\vec{d}$.
- We call
$$\vec{p} = \vec{p}_0 + t\vec{d}$$
the *vector equation* of the line.
- Note: if the line ℓ passes through $\vec{p}_0$ and $\vec{q}_0$, then the vector equation of ℓ is
$$\vec{p} = \vec{p}_0 + t(\vec{q}_0 - \vec{p}_0).$$

Scalar equations

- If ℓ has vector equation $\vec{p} = \vec{p}_0 + t\vec{d}$, where
$$\vec{p} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \vec{p}_0 = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}, \text{ and } \vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix},$$
then
$$\begin{aligned} x_1 &= p_1 + td_1, \\ x_2 &= p_2 + td_2, \\ \vdots &= \quad \vdots \\ x_n &= p_n + td_n. \end{aligned}$$
- We call these equations the *scalar equations*, or *parametric equations*, of ℓ.

Example

- To find the equation of the line ℓ through the points

$$\vec{q}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \vec{p}_0 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix},$$

we first find

$$\vec{d} = \vec{q}_0 - \vec{p}_0 = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}.$$

- Then the vector equation of the line is

$$\vec{p} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}.$$

Example (cont'd)

- The parametric equations of the line are

$$\begin{aligned} x &= 2 - t \\ y &= -1 + 3t \\ z &= -1 + 4t. \end{aligned}$$

Example

- Suppose we wish to find the distance D from the point

$$\vec{q} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

to the line ℓ with equation

$$\vec{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

- Let

$$\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \vec{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Example (cont'd)

- Then if we let

$$\vec{v}_1 = \operatorname{proj}_{\vec{d}} \vec{v} = \frac{6}{5}\vec{d} = \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix},$$

the desired distance is the length of

$$\vec{v}_2 = \vec{v} - \vec{v}_1 = \begin{bmatrix} -\frac{7}{5} \\ \frac{14}{5} \end{bmatrix}.$$

- Hence

$$D = \frac{7}{5}\sqrt{5}.$$

- Note: the point on ℓ closest to $\vec{q}$ is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix} = \begin{bmatrix} \frac{17}{5} \\ \frac{11}{5} \end{bmatrix}.$$

Normal form in $\mathbb{R}^2$

- Let ℓ be a line in $\mathbb{R}^2$ with vector equation $\vec{p} = \vec{p}_0 + t\vec{d}$.
- Let $\vec{n}$ be a nonzero vector orthogonal to $\vec{d}$.
- For example, if

$$\vec{d} = \begin{bmatrix} u \\ v \end{bmatrix},$$

  then we could take

$$\vec{n} = \begin{bmatrix} -v \\ u \end{bmatrix}.$$

  cd
- Note: if $\vec{p}$ is a point on ℓ, then $\vec{p} - \vec{p}_0 = t\vec{d}$ for some scalar t, and so $\vec{p} - \vec{p}_0$ is orthogonal to $\vec{n}$.
- Note:
  - If $\vec{p}$ is a point in $\mathbb{R}^2$ such that $\vec{p} - \vec{p}_0$ is orthogonal to $\vec{n}$, then $\vec{p} - \vec{p}_0$ satisfies the equation $\vec{n} \cdot \vec{x} = \vec{0}$.
  - Since $\vec{d}$ is the only basic solution to this equation, it follows that $\vec{p} - \vec{p}_0 = t\vec{d}$ for some scalar t.
  - That is, $\vec{p}$ is on ℓ.

Normal form in $\mathbb{R}^2$ (cont'd)

- Conclusion: $\vec{p}$ is on the line ℓ if and only if $\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0$.
- We call this the *normal equation* of ℓ.
- Note: if we write

$$\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}, \vec{p} = \begin{bmatrix} x \\ y \end{bmatrix},$$

  and let $c = -\vec{n} \cdot \vec{p}_0$, then we may write the normal equation of ℓ as

$$ax + by + c = 0.$$

Example

- Suppose ℓ passes through the points

$$\vec{p}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \vec{q}_0 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

- Let

$$\vec{d} = \vec{q}_0 - \vec{p}_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

- Then

$$\vec{n} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

  is orthogonal to $\vec{d}$.
- So the normal equation for ℓ is

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = 0.$$

Example (cont'd)

- That is,

$$-3(x - 1) + (y - 2) = 0.$$

- That is,

$$-3x + y = -1.$$