Mathematics 160: Lecture 22

Planes

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Planes in $\mathbb{R}^n$

- Suppose $\vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^n$ which are not parallel.
- Given any third vector $\vec{p}_0$ in $\mathbb{R}^n$, we call the set of all points $\mathcal{P}$

  $$\vec{p} = \vec{p}_0 + s\vec{v} + t\vec{w},$$

  where s and t are scalars, a *plane*.
- We call this equation a *vector equation* for $\mathcal{P}$.

Parametric equations

- If

  $$\vec{p} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \vec{p}_0 = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \text{ and } \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix},$$

  then

  $$\begin{aligned} x_1 &= p_1 + sv_1 + tw_1, \\ x_2 &= p_2 + sv_2 + tw_2, \\ \vdots &= \vdots \\ x_n &= p_n + sv_n + tw_n, \end{aligned}$$

  which are the *scalar*, or *parametric*, *equations* for $\mathcal{P}$.

Example

- Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ which contains the points

  $$\vec{p}_0 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{q} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \text{ and } \vec{r} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

- Then

  $$\vec{v} = \vec{q} - \vec{p}_0 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \text{ and } \vec{w} = \vec{r} - \vec{p}_0 = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

  are vectors which lie on $\mathcal{P}$.

Example (cont'd)

- So the vector equation of $\mathcal{P}$ is

$$\vec{p} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + t\begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix},$$

and the parametric equations are

$$\begin{aligned} x &= 1 - 2t, \\ y &= -1 + 3s + 2t, \\ z &= 1 + s + 2t. \end{aligned}$$

Normal form in $\mathbb{R}^3$

- Suppose $\mathcal{P}$ is a plane in $\mathbb{R}^3$ with vector equation $\vec{p} = \vec{p}_0 + s\vec{v} + t\vec{w}$.
- Analogous to the case with lines in $\mathbb{R}^2$, if $\vec{n}$ is perpendicular to both $\vec{v}$ and $\vec{w}$, then we may describe $\mathcal{P}$ as the set of all $\vec{p}$ in $\mathbb{R}^3$ satisfying $\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0$.
- This is the *normal equation* of $\mathcal{P}$.

Normal vectors

- Given

$$\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix},$$

we want to find a vector $\vec{n}$ which is orthogonal to both $\vec{v}$ and $\vec{w}$.
- Note:

$$\begin{aligned} 0 &= \det \begin{bmatrix} x_1 & x_1 & x_2 \\ y_1 & y_1 & y_2 \\ z_1 & z_1 & z_2 \end{bmatrix} \\ &= x_1(y_1z_2 - y_2z_1) - y_1(x_1z_2 - x_2z_1) + z_1(x_1y_2 - x_2y_1). \end{aligned}$$

Normal vectors (cont'd)

- And:

$$\begin{aligned} 0 &= \det \begin{bmatrix} x_2 & x_1 & x_2 \\ y_2 & y_1 & y_2 \\ z_2 & z_1 & z_2 \end{bmatrix} \\ &= x_2(y_1z_2 - y_2z_1) - y_2(x_1z_2 - x_2z_1) + z_2(x_1y_2 - x_2y_1). \end{aligned}$$

- Hence

$$\vec{n} = \begin{bmatrix} y_1z_2 - y_2z_1 \\ x_2z_1 - x_1z_2 \\ x_1y_2 - x_2y_1 \end{bmatrix}$$

is orthogonal to both $\vec{v}$ and $\vec{w}$.

Definition

- Given vectors
$$\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$
in $\mathbb{R}^3$, we call the vector
$$\vec{v} \times \vec{w} = \begin{bmatrix} y_1 z_2 - y_2 z_1 \\ x_2 z_1 - x_1 z_2 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$
the *cross product* of $\vec{v}$ and $\vec{w}$.
- Note: $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.

Computation

- Let
$$\vec{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$
- If
$$\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix},$$
then
$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{\imath} & x_1 & x_2 \\ \vec{\jmath} & y_1 & y_2 \\ \vec{k} & z_1 & z_2 \end{bmatrix}.$$
- Note: it follows that $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

Example

- For our earlier example,
$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{\imath} & 0 & -2 \\ \vec{\jmath} & 3 & 2 \\ \vec{k} & 1 & 2 \end{bmatrix} = 4\vec{\imath} - 2\vec{\jmath} + 6\vec{k} = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}.$$
- Hence the normal form of the equation for $\mathcal{P}$ is
$$\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) = 0.$$
- Or, equivalently,
$$4x - 2y + 6z = 12.$$

Distance to a plane

- Let $\mathcal{P}$ be a plane in $\mathbb{R}^3$ with normal equation $\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0$. Let D be the shortest distance from a point $\vec{q}$ to $\mathcal{P}$.
- Let $\vec{v} = \vec{q} - \vec{p}_0$.
- Note: the component of $\vec{v}$ in the direction of $\vec{n}$ is
$$\frac{\vec{n} \cdot \vec{v}}{\|\vec{n}\|} = \frac{\vec{n} \cdot \vec{q} - \vec{n} \cdot \vec{p}_0}{\|\vec{n}\|}.$$
- Hence
$$D = \frac{|\vec{n} \cdot \vec{q} - \vec{n} \cdot \vec{p}_0|}{\|\vec{n}\|}.$$
- Note: the point on $\mathcal{P}$ closest to $\vec{q}$ is $\vec{q} - \text{proj}_{\vec{n}} \vec{v}$.

Example

- Suppose $\mathcal{P}$ is a plane with equation $4x + 2y + 4z = 8$ and $\vec{q} = \begin{bmatrix} 3 & 3 & 4 \end{bmatrix}^T$.
- Note: $\vec{n} = \begin{bmatrix} 4 & 2 & 4 \end{bmatrix}^T$ is a normal vector for $\mathcal{P}$.
- Moreover, for any point $\vec{p}_0$ on $\mathcal{P}$, $\vec{n} \cdot \vec{p}_0 = 8$.
- Hence the shortest distance D from $\vec{q}$ to $\mathcal{P}$ is

$$D = \frac{|\vec{n} \cdot \vec{q} - 8|}{\|\vec{n}\|} = \frac{|34 - 8|}{6} = \frac{13}{3}.$$

Example (cont'd)

- Note: If $\vec{p}_0$ is a point on $\mathcal{P}$ and $\vec{v} = \vec{q} - \vec{p}_0$, then

$$\text{proj}_{\vec{n}}\, \vec{v} = \frac{26}{36} \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} = \frac{13}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix},$$

so the point on $\mathcal{P}$ closest to $\vec{q}$ is

$$\vec{q} - \text{proj}_{\vec{n}}\, \vec{v} = \begin{bmatrix} \frac{1}{9} \\ \frac{14}{9} \\ \frac{10}{9} \end{bmatrix}.$$

Octave commands

- If `u` and `v` are either both 3×1 or both 1×3 matrices, then `cross(u, v)` will compute the cross product of u and v.
- If `u` and `v` are either both column matrices or both row matrices, then `dot(u, v)` will compute the dot product of u and v.