Mathematics 160: Lecture 23

Linear Transformations

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Linear transformations

- We often call a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ a *transformation*.
- We say a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is *linear* if, for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ and all scalars α,
  - $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$, and
  - $T(\alpha\vec{v}) = \alpha T(\vec{v})$.

Example

- Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

- Note: T reflects a vector about the x-axis.
- If $\vec{v} = \begin{bmatrix} x_1 & y_1 \end{bmatrix}^T$ and $\vec{w} = \begin{bmatrix} x_2 & y_2 \end{bmatrix}^T$, then

$$\begin{aligned} T(\vec{v} + \vec{w}) &= T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -y_1 - y_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 \\ -y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ -y_2 \end{bmatrix} = T(\vec{v}) + T(\vec{w}). \end{aligned}$$

Example (cont'd)

- If $\vec{v} = \begin{bmatrix} x & y \end{bmatrix}^T$ and α is a scalar, then

$$T(\alpha\vec{v}) = T\left(\begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix}\right) = \begin{bmatrix} \alpha x \\ -\alpha y \end{bmatrix} = \alpha \begin{bmatrix} x \\ -y \end{bmatrix} = \alpha T(\vec{v}).$$

- Hence T is a linear transformation.
- Note: if we let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then $T(\vec{v}) = A\vec{v}$.

Matrix multiplication

- Given a 2×2 matrix A, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{v}) = A\vec{v}$.
- Then, for any vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ and any scalar α,

$$T(\vec{v} + \vec{w}) = A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = T(\vec{v}) + T(\vec{w})$$

  and

$$T(\alpha\vec{v}) = A(\alpha\vec{v}) = \alpha A\vec{v} = \alpha T(\vec{v}).$$

- Hence T is a linear transformation.
- Note:
  - Let

$$\vec{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

    be the *coordinate vectors* for $\mathbb{R}^2$.
  - Then

$$\begin{aligned} T(\vec{\imath}) &= A\vec{\imath} = \text{ first column of } A \\ T(\vec{\jmath}) &= A\vec{\jmath} = \text{ second column of } A \end{aligned}$$

The matrix of a linear transformation

- Now suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.
- Let A be the matrix with columns $T(\vec{\imath})$ and $T(\vec{\jmath})$.
- Note: if $\vec{v} = \begin{bmatrix} x & y \end{bmatrix}^T$ is any vector in $\mathbb{R}^2$, then

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x\vec{\imath} + y\vec{\jmath}.$$

- It follows that

$$T(\vec{v}) = T(x\vec{\imath} + y\vec{\jmath}) = xT(\vec{\imath}) + yT(\vec{\jmath}) = A\vec{v}.$$

- Hence we have shown that
  - A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear if and only if there exists a 2×2 matrix A such that $T(\vec{v}) = A\vec{v}$ for all $\vec{v}$ in $\mathbb{R}^2$.
  - Moreover, the columns of A are $T(\vec{\imath})$ and $T(\vec{\jmath})$.

Example

- Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which rotates a vector in $\mathbb{R}^2$ through an angle θ.
- Note:

$$R_\theta(\vec{\imath}) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \text{ and } R_\theta(\vec{\jmath}) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}.$$

- It follows that, if R_θ is linear, then $R_\theta(\vec{v}) = A\vec{v}$ where

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Example (cont'd)

- Now for any $\vec{v} = \begin{bmatrix} x & y \end{bmatrix}^T$, we have

$$A\vec{v} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix}.$$

- It is easy to see than that $A\vec{v} \cdot A\vec{v} = x^2 + y^2$ and $A\vec{v} \cdot \vec{v} = (x^2 + y^2)\cos(\theta)$.
- Hence $\|A\vec{v}\| = \|\vec{v}\|$ and the cosine of the angle between $\vec{v}$ and $A\vec{v}$ is

$$\frac{A\vec{v} \cdot \vec{v}}{\|A\vec{v}\|\|\vec{v}\|} = \cos(\theta).$$

- That is, $R_\theta(\vec{v}) = A\vec{v}$.

Compositions

- Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ are both linear transformations, with associated matrices A and B, respectively. Then, for any $\vec{v}$ in $\mathbb{R}^2$,

$$(T \circ S)(\vec{v}) = T(S(\vec{v})) = T(B\vec{v}) = A(B\vec{v}) = AB\vec{v}.$$

- That is, the matrix corresponding to the linear transformation $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$ is AB.

Example

- Suppose T reflects a vector about the x-axis and S rotates a vector through an angle π.
- Then

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

- So

$$(T \circ S) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

- Hence $T \circ S$ is a reflection about the y-axis.

Inverses

- We say a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is *invertible* if there exists a linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for any $\vec{v}$ in $\mathbb{R}^2$,

$$(S \circ T)(\vec{v}) = \vec{v} \text{ and } (T \circ S)(\vec{v}) = \vec{v}.$$

- We let T^{-1} denote the inverse of T.
- Note: if A is the matrix for T and B is the matrix for T^{-1}, then we must have $BA = I$ and $AB = I$.
- That is, if A is the matrix of T, then A^{-1} is the matrix of T^{-1}.
- In particular, T is invertible if and only if its matrix A is invertible.

Example

- Let R_θ be the linear transformation which rotates a vector through an angle θ.
- Clearly, we should have $R_\theta^{-1} = R_{-\theta}$.
- Hence if

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

we should have

$$A^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix},$$

which may be verified easily.